Dimensional Analysis & similarities:-

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Introduction:- It is mathematical technique used in research work for design and for conducting model tests. It deals with the dimensions of the physical quantities involved in the phenomenon.

→ All physical quantities are measured by comparison, which is made with respect to an arbitrarily fixed value.

→ Length (L), mass (M) and time (T) are three fixed dimensions which are of importance in Fluid Mechanics.

Primary Dimension	Symbol	SI Unit
1) mass	M	kg
2) Length	L	m
3) time	T	sec
4) temperature	θ	K (kelvin)
5) electric current	I	A (ampere)
6) amount of light (luminous intensity)	C	c (candela)
7) amount of matter	N or N	mol (mole)

→ These fixed dimensions are called fundamental dimension or fundamental quantity or Primary dimensions

Dimensional Homogeneity

Dimensional homogeneity means the dimensions of each terms in an equation on both sides are equal. Thus if the dimensions of each term on both sides of an equation are the same the equation is known as dimensionally homogeneous equation. The powers of fundamental dimensions on both side of the equation will be identical for a dimensionally homogeneous equation

$$V = \sqrt{2gH}$$

Dimension of L.H.S $V = \frac{L}{T} = LT^{-1}$

Dimension of R.H.S $\sqrt{2gH} = \sqrt{\frac{L}{T^2} \times L} = \sqrt{\frac{L^2}{T^2}} = LT^{-1}$

eqn. $V = \sqrt{2gH}$ is ~~densionally~~ dimensionally homogeneous

Methods of Dimensional Analysis

1) Rayleigh's method
2) Buckingham's π-theorem

1) Rayleigh's method:-

This method is used for determining the expression for a variable which depends upon maximum three or four variables only

Let X is a variable, which depends on X_1, X_2 and X_3 variables, Then according to Rayleigh's method X is function of X_1, X_2 and X_3 and mathematically it is written as $X = f[X_1, X_2, X_3]$

This can also be written as $X = K X_1^a \cdot X_2^b \cdot X_3^c$

where K is constant and a, b, & c are arbitrary powers.

The value of a, b, and c are obtain by comparing the power of the fundamental dimension on both sides

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2) Buckingham's π-Theorem:

Buckingham's π-Theorem states, "If there are **n** variables (independent and dependent variables) in a physical phenomenon and if these variables contain **m** fundamental dimensions (M, L, T), Then the variables are are arranged in to (n−m) dimensionless terms. Each term is called π-term"

Let $X_1, X_2, X_3, \ldots X_n$ are the variables involved in a physical problem. Let X_1 be the dependent variable and $X_2, X_3, X_4 \ldots X_n$ are the independent variables on which X_1 depends. Then X_1 is a function of $X_2, X_3, \ldots X_n$ and mathematically it is expressed

$$X_1 = f(X_2, X_3 \ldots X_n) \quad \ldots (1)$$

Eq^n (1) can also be written as

$$f_1(X_1, X_2, X_3 \ldots X_n) = 0 \quad \ldots (2)$$

Eq^n. (2) is a dimensionally homogeneous equation. It contains n variables. If there are m fundamental dimensions then according to Buckingham's π-theorem, eq^n (2) can be written in terms of number of dimensionless groups or π-terms in which number of π-terms is equal to $(n-m)$. Hence eq^n. (2) becomes as

$$f(\pi_1, \pi_2, \ldots \pi_{n-m}) = 0 \quad \ldots\ldots (3)$$

Each of π-terms is dimensionless and is independent of the system. Division or multiplication by a constant does not change the charactor of the π-term. Each π-term contains $m+1$ variables, where m is the number of fundamental dimensions and is also called repeating variables. Let in the above case X_2, X_3 and X_4 are repeating variables if the fundamental dimension $m\ (M, L, T) = 3$. Then each π-term is written as

$$\left.\begin{aligned} \pi_1 &= X_2^{a_1} \cdot X_3^{b_1} \cdot X_4^{c_1} \cdot X_1 \\ \pi_2 &= X_2^{a_2} \cdot X_3^{b_2} \cdot X_4^{c_2} \times X_5 \\ &\vdots \\ \pi_{n-m} &= X_2^{a_{n-m}} \cdot X_3^{b_{n-m}} \cdot X_4^{c_{n-m}} \cdot X_n \end{aligned}\right\} \quad \ldots \quad (4)$$

Each eq^n. is solved by the principle of dimensional homogeneity and values of a_1, b_1, c_2 etc., are obtained. These values are subtituted in eq^n (4) and values of $\pi_1, \pi_2 \ldots \pi_{n-m}$ are obtained. These values of π's are subtituted in eq^n (3)

$$\pi_1 = \phi\,[\pi_2, \pi_3 \ldots \pi_{n-m}]$$

$$\pi_2 = \phi\,[\pi_1, \pi_3 \ldots \pi_{n-m}]$$

Method of Selecting Repeating Variables.

The number of repeating variables are equal to the number of fundamental dimensions of the problem.

1) As far as possible, the dependent variable should not be selected as repeating variable.

2) The repeating variables should be choosen in such a way that one variable contains geometric property, other variable contains flow property and third variable contains fluid property.

Variables with Geometric Property are
(i) Length, L (ii) diameter, d (iii) Height, H ect

Variables with flow property are
(i) Velocity, V (ii) Acceleration ect

Variable with fluid property
(i) μ, viscosity (ii) Density, ρ (iii) kinematic viscosity, ν

3) The repeating variables selected should not form a dimensionless group

4) The repeating variables together must have the same number of fundamental dimensions.

5) No two repeating variables should have the same dimensions.

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Similitude - Type of Similarities

similitude is defined as the similarity between the model and prototype in every respect, which means that the model and prototype have similar properties or model and prototype are completely similar.

Three type of similarities must exist between the model and prototype

1) Geometric similarity
2) Kinematic Similarity
3) Dynamic similarity

1) Geometric similarity

The geometric similarity is said to exist between the model and the prototype. The ratio of all corresponding linear dimension in the model and prototype are equal.

Let L_m = Length of model
b_m = Breadth of model
D_m = Diameter of model
A_m = Area of model
V_m = Volume of model

and L_p, b_p, D_p, A_p, V_p = Corresponding value of the prototype

For geometric similarity bet^n model and prototype

$$\frac{L_p}{L_m} = \frac{b_p}{b_m} = \frac{D_p}{D_m} = L_r$$

where L_r is called the scale ratio

Area $\frac{A_p}{A_m} = \frac{L_p \times b_p}{L_m \times b_p} = L_r \times L_r = L_r^2$ Volume $\frac{V_p}{V_m} = L_r^3$

2) Kinematic Similarity :-

Kinematic similarity means the similarity of motion between model and prototype. Thus kinematic similarity is said to exist betn. the model and the prototype if the ratios of velocity and acceleration at the corresponding point in the model and at the corresponding point in the prototype are the same.

V_{P_1} = Velocity of fluid at point 1 in prototype

V_{P_2} = " " " " " 2 "

a_{P_1} = Acceleration of fluid at point 1 in prototype

a_{P_2} = " " " " 2 "

$V_{m_1}, V_{m_2}, a_{m_1}, a_{m_2}$ = corresponding value at the corresponding points of fluid velocity and acceleration in model

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Kinematic similarity $\dfrac{V_{P_1}}{V_{m_1}} = \dfrac{V_{P_2}}{V_{m_2}} = V_r$

Where V_r is the velocity ratio

For acceleration

$$\frac{a_{P_1}}{a_{m_1}} = \frac{a_{P_2}}{a_{m_2}} = a_r$$

Where a_r is the acceleration ratio

3) Dynamic Similarity:-

Dynamic similarity means the similarity of forces bet^n. the model and the prototype. Thus dynamic similarity is said to exist between the model and the prototype if the ratios of the corresponding forces acting at the corresponding points are equal.

Dynamic similarity

$$\frac{(F_i)_p}{(F_i)_m} = \frac{(F_v)_p}{(F_v)_m} = \frac{(F_g)_p}{(F_g)_m} = F_r$$

where F_r is the force ratio

Dimensionless Number

1) **Reynold's Number:-** It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid.

$$Re = \frac{\rho V L}{\mu}$$ where L characteristic length

2) **Froude's Number:-** The froude's number is defined as the square root of the ratio of inertia force of flowing fluid to the gravity force.

$$F_e = \sqrt{\frac{F_i}{F_g}} = \frac{V}{\sqrt{Lg}}$$

3) **Euler's Number (Eu):-** It is defined as the square root of the ratio of the inertia force of a flowing fluid to the pressure force.

$$Eu = \sqrt{\frac{F_i}{F_p}} = \frac{V}{\sqrt{P/\rho}}$$

4) Weber's Number (We)

It is defined as the square root of the ratio of the inertia force of a flowing fluid to the surface tension force.

$$We = \sqrt{\frac{F_i}{F_s}} = \frac{V}{\sqrt{\sigma / \rho L}}$$

5) Mach's Number (M)

Mach's number is defined as the square root of the ratio of the inertia force of a flowing fluid to the elastic force

$$M = \sqrt{\frac{F_i}{F_e}} = \frac{V}{\sqrt{K/\rho}} = \frac{V}{C}$$

C = Velocity of sound or acoustic

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→ **Model Laws :-**

For dynamic similarity between the model and the prototype, the ratio of the corresponding forces ~~are~~ ~~dime~~ acting at the corresponding point in the model and prototype should be equal. The ratio of the forces are dimensionless numbers. It means for dynamic similarity betn the model and prototype, the dimensionless number should be same for model and the prototype. But it is quite difficult to satisfy the condition that all the dimensionless number (i.e. Re, Fe, We, Eu & M) are the same for the model and prototype. Hence models are designed on the basis of ratio of the force, which is dominating in the phenomenon.

The laws on which the models are designed for dynamic similarity are called model Laws or Laws of similarity

1) Reynold's Model Law

Reynold's model Law is applied in following fluid flow problems:-

(i) Pipe flow

(ii) Resistance experienced by sub-marines, airplanes, fully immersed bodies

$$[Re]_m = [Re]_p$$

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p} \Rightarrow \cancel{\frac{\rho_m}{\rho_p} \cdot \frac{V_m}{V_p}}$$

$$\frac{\rho_p}{\rho_m} \cdot \frac{V_p}{V_m} \cdot \frac{L_p}{L_m} \times \frac{1/\mu_p}{1/\mu_m} = 1 \quad \text{or} \quad \frac{\rho_r V_r L_r}{\mu_r} = 1$$

The scale ratio for time, acceleration, force and discharge for Reynold's model Law

$$t_r = \text{Time scale ratio} = \frac{L_r}{V_r} \quad \left\{ \because V = \frac{L}{t} \right\}$$

$$a_r = \text{Acceleration scale ratio} = \frac{V_r}{t_r}$$

$$F_r = \text{Force scale ratio} = [\text{Mass} \times \text{accle.}]_r$$

$$= m_r \times a_r = \rho_r A_r V_r \times a_r$$

$$= \rho_r L_r^2 V_r \times a_r$$

$$Q_r = \text{Dischare scale ratio} = (\rho A V)_r$$

$$= \rho_r A_r V_r = \rho_r \cdot L_r^2 \cdot V_r$$

2) Froude Model Law:-

Froude model Law is applicable when the gravity force is only predominant force which controls the flow in addition to the force of inertia.

Froude model Law is applied in the following fluid flow problems:-

i) Free surface flows such as flow over spillways, weirs, sluices, channels ect.

ii) Flow of jet from an orifice or Nozzle.

iii) Where waves are likely to be formed on surface,

(iv) Where fluid of different densities flow over one another.

$$(Fe)_{model} = (Fe)_{prototype} \Rightarrow \frac{V_m}{\sqrt{g_m L_m}} = \frac{V_p}{\sqrt{g_p L_p}}$$

$$g_m = g_p \quad [\because \text{ same place}]$$

$$\frac{V_m}{\sqrt{L_m}} = \frac{V_p}{\sqrt{L_p}} \Rightarrow \frac{V_m}{V_p} \times \frac{1}{\sqrt{\frac{L_m}{L_p}}} = 1$$

$$\frac{V_p}{V_m} = \sqrt{\frac{L_p}{L_m}} = \sqrt{L_r} \Rightarrow V_r = \sqrt{L_r}$$

a) Scale ratio for time

$$t_r = \frac{t_p}{t_m} = \frac{(L/V)_p}{(L/V)_m} = \frac{L_p}{L_m} \times \frac{V_m}{V_p} = L_r \times \frac{1}{\sqrt{L_r}}$$

$$= \sqrt{L_r}$$

b) Scale ratio for acceleration $\quad a = V/t$

$$a_r = \frac{a_p}{a_m} = \frac{(V/t)_p}{(V/t)_m} = \frac{V_p}{V_m} \times \frac{t_m}{t_p} = \sqrt{L_r} \times \frac{1}{\sqrt{L_r}} = 1$$

c) Scale ratio for discharge:-

$$Q = A \cdot V = L^2 \times \frac{L}{t} = \frac{L^3}{t}$$

$$Q_r = \frac{Q_P}{Q_m} = \frac{(L^3/t)_P}{(L^3/t)_m} = \left(\frac{L_P}{L_m}\right)^3 \times \frac{t_m}{t_P}$$

$$= \frac{L_P^3}{L_m^3} \times \frac{1}{\sqrt{L_r}} = L_r^3 \times \frac{1}{\sqrt{L_r}}$$

$$= L_r^{2.5}$$

d) scale ratio for force

$$\text{Force} = \text{mass} \times \text{acceleration} = \rho L^3 \times \frac{V}{t} = \rho L^2 \frac{L}{t} \cdot V$$

$$= \rho L^2 V^2$$

$$F_r = \frac{F_P}{F_m} = \frac{\rho_P L_P^2 V_P^2}{\rho_m L_m^2 V_m^2} = \frac{\rho_P}{\rho_m} \times \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2$$

If the fluid used in model and prototype is same then

$$\rho_m = \rho_P$$

$$F_r = \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2 = L_r^2 \times \left(\sqrt{L_r}\right)^2 = L_r^3$$

e) scale ratio for work, energy, torque, moment

$$\text{Torque} = \text{Force} \times \text{Distance} = F \times L$$

$$\text{Torque ratio } T_r^* = \frac{T_P^*}{T_m^*} = \frac{(F \times L)_P}{(F \times L)_m} = F_r \times L_r$$

$$= L_r^3 \times L_r$$

$$= L_r^4$$

(f) scale ratio for power

$$\text{power} = \text{work per unit time}$$

$$= \frac{F \times L}{t}$$

$$P_r = \frac{P_P}{P_m} = \frac{(F \times L/t)_P}{(F \times L/t)_m} = \frac{F_P}{F_m} \times \frac{L_P}{L_m} \times \frac{t_m}{t_P} = L_r^3 \times L_r \times \frac{1}{\sqrt{L_r}}$$

$$= L_r^{3.5}$$

3) Euler's Model Law

Euler's model Law is applicable when the pressure forces are alone predominant in addition to the inertia force

$$(Eu)_{model} = (Eu)_{prototype}$$

$$\frac{V_m}{\sqrt{P_m/\rho_m}} = \frac{V_p}{\sqrt{P_p/\rho_p}}$$

If fluid is same $\rho_m = \rho_p$

$$\frac{V_m}{\sqrt{P_m}} = \frac{V_p}{\sqrt{P_p}}$$

Euler's model Law is applied for fluid flow problems where flow is taking place in a closed pipe in which case turbulence is fully developed so that viscous forces are negligible and gravity force and surface tension force is absent. This Law is also used where the phenomenon of cavitation takes place.

4) Weber Model Law :-

$$(We)_{model} = (We)_{prototype}$$

$$\frac{V_m}{\sqrt{\sigma_m/\rho_m L_m}} = \frac{V_p}{\sqrt{\sigma_p/\rho_p L_p}}$$

1) Capillary rise in narrow passages
2) Capillary movement of water in soil
3) Capillary waves in channels
4) Flow over weirs for small heads.

5) Mach Model Law

$$(M)_{model} = (M)_{prototype}$$

$$\frac{V_m}{\sqrt{K_m/\rho_m}} = \frac{V_p}{\sqrt{K_p/\rho_p}}$$

Mach model Law is applied in following cases

1) Flow of aeroplane and projectile through air at supersonic speed.
2) Aerodynamic testing.
3) Under water testing of torpedoes
4) Water-hammer problems.

Classification of Models:-

1) Undistorted Model 2) Distorted Model

1) **Undistorted Model:-** Undistorted models are those models which are geometrically similar to their prototypes or in other words if the scale ratio for the linear dimensions of the model and its prototype is same, the model is called undistorted model.

2) **Distorted Models.** A model is said to be distorted if it is not geometrically similar to its prototype. For a distorted model different scale ratios for the linear dimensions are adopted. For example, in case of rivers, harbours reservoirs ect. two different scale ratios, one for horizontal dimensions and other for vertical dimensions are taken. Thus the models of rivers, harbours and reservoirs will become as distorted models. if for the river, the horizontal and vertical scale ratio are taken to be same so that the model is undistorted, then depth of water in model of the river will be very-very small which may not be measured accurately