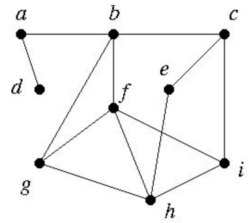
Graph Theory

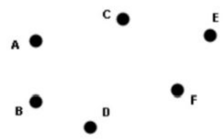
Graph: A graph G = (V,E) is a mathematical structure consisting of two sets, a non empty set V known as the vertex set and a set E known as the edge set.



The elements of V and E are called vertices and edges respectively. Precisely V and E are also denoted as $V(G)$ and $E(G)$ respectively.

Trivial Graph: A graph with single vertex and no edges is known as trivial graph.

Null Graph: A graph for which edge set is empty is called null graph.



Loop: An edge whose both the end points are same is known as a loop.

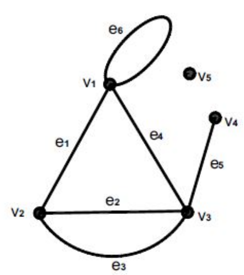
Isolated vertex: A vertex which is not the end vertex of any edge is known as an isolated vertex.

Parallel Edges or Multiple Edges: If two or more edges have the same end vertices are known as parallel edges or multiple edges.

Simple Graph: A graph without loops or parallel edges is called simple graph.

Adjacent vertices: The vertices which are joined by an edge are known as adjacent vertices and the adjacent vertices are called neighbors.

Here edge e_6 is a loop, e_2 and e_3 are multiple edges, v_5 is an isolated vertex.



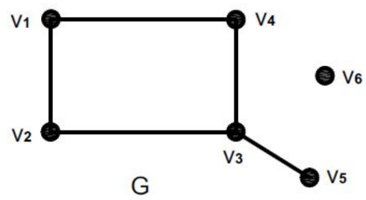
Incidence of a vertex and edge: If a vertex v is an end point of edge e then v is said to be incident on e. e and v are called incident to each other.

Degree of a vertex: A degree of a vertex v in a graph G is defined as number of edges incident on v plus twice the number of loops. The degree of a vertex v is denoted by $d(v)$ or $d_G(v)$.

Pendent vertex: A vertex of degree 1 is called a pendent vertex.

Odd & Even vertex: A vertex of a graph is called odd or even depending on whether its degree is odd or even.

For the graph G shown in the following figure, we have $d(v_1) = d(v_2) = d(v_4) = 2$ $d(v_3) = 3$ $d(v_5) = 1$ and $d(v_6) = 0$.



Theorem: For any graph G with e edges and n vertices $v_1, v_2, \ldots, v_n$ the sum of degrees of vertices of a graph is twice the number of edges.

Proof: Since each edge has two end points, each edge will precisely contribute 2 to the sum of degrees. Moreover each loop will also contribute 2 to the sum of degrees.

Hence, $\sum_{i=1}^{n} d(v_i) = 2e$.

Theorem: For any graph G there is an even number of odd vertices.

Proof: Let A and B be the set of even vertices and odd vertices respectively of graph G. Then for each $a \in A$, $d(a)$ is even. This implies that $\sum_{a \in A} d(a)$ is even.

Now by first theorem of graph theory we have

$$\sum_{a \in A} d(a) + \sum_{b \in B} d(b) = 2e.$$

Thus $\sum_{b \in B} d(b) = 2e - \sum_{a \in A} d(a)$ which is even being difference of two even numbers.

As all the terms in $\sum_{b \in B} d(b)$ are odd, the number of elements in B must be even. Thus for G there is an even number of odd vertices.

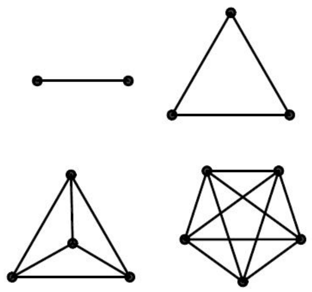
Regular graph: A graph is said to be regular if all of its vertices have equal degree.

If for every vertex v of a graph G, $d(v) = k$ then that graph is known as k-regular.

Complete graph: A complete graph is a simple graph in which every pair of vertices is joined by an edge.

We also note that a complete graph with n vertices is $(n-1)$-regular. A complete graph on n vertices is denoted by K_n.

In the following figure, the complete graphs on two, three, four and five vertices are shown.

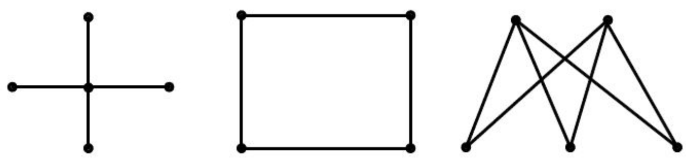


Bipartite graph: Let G be a graph with vertex set V. If V can be partitioned into two subsets such that $V = V_1 \cup V_2$ and each edge of G has one end vertex in V_1 and other end vertex in V_2 then V is called bipartition of G and G is called a bipartite graph.

Complete Bipartite graph: A complete bipartite graph is a simple bipartite graph with bipartition $V = V_1 \cup V_2$ such that every vertex in V_1 is joined to every vertex of V_2.

If V_1 has m-vertices and V_2 has n-vertices then such complete bipartite graph is denoted by $K_{m,n}$. The complete bipartite graph $K_{1,n}$ is known as a Star graph.

Here some complete bipartite graphs are shown.

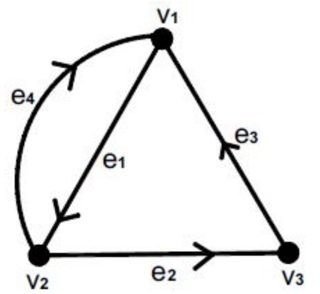


$K_{1,4}$ $K_{2,2}$ $K_{2,3}$

A directed edge (or arc) is an edge whose one end vertex is designated as the 'tail' and other end vertex is designated as the 'head'.

An arc is said to be directed from 'tail' to 'head'. A multiarc or multiple arc is a set of two or more arcs having the same head and tail.

A graph whose every edge is directed is called a directed graph or digraph.



A graph which contains directed as well as undirected edges is called partially directed graph.

The underlying graph of directed or partially directed graph G is the graph resulted by removing all the designations 'head' and 'tail' from the graph G.

The indegree of a vertex v in a digraph is the number of arcs directed to v and the outdegree of a vertex v is the number of arcs directed away from v.

Each loop at v counts one towards indegree and one towards outdegree.

Theorem: In a digraph, the sum of indegrees and outdegrees are both equal to the number of edges.

Proof: Each directed edge contributes one to the indegree at 'head' and one to the outdegree at 'tail'.

Subgraph: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph H with vertex set $V(H)$ and edge set $E(H)$ is said to be subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and each edge of H has same end vertices as in G.

Every graph G is a subgraph of itself. The subgraph of G other than G is called a proper subgraph.

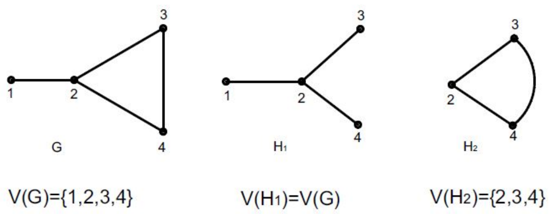
If H_2 is a subgraph of H_1 and H_1 is a subgraph of G then H_2 is also a subgraph of G. A single vertex of a graph G is a subgraph of G. A single edge of a graph G is a subgraph of G.

Induced subgraph: In a graph G the induced subgraph on a set of vertices $H = h_1, h_2, \ldots, h_k$ denoted as $G(H)$ has H as its vertex set and it contains every edge of G whose end vertices are in H i.e. $V(G(H)) = H$ and $E(G(H)) = \{e \in E(G)$ / the end vertices of e are in $H\}$

In other words if subgraph H satisfies the added property that $uv \in E(G(H))$ if and only if $uv \in E(G)$ then H is a induced subgraph of G.

Spanning subgraph: A subgraph H of a graph G is a spanning subgraph if $V(H) = V(G)$.
For the graph G , H_1 is a spanning subgraph but not an induced subgraph as there is an edge between 3 and 4 in G but it is not in H_1.

Where H_2 is an induced subgraph but not a spanning subgraph.



Walk: A walk of a graph G is a finite alternating sequence of vertices and edges. If W is a walk between the vertices v0 and vk then we denote it as $v - v_k$. Here v_0 and v_k are called origin and terminus while remaining vertices are called internal vertices.

The number of edges in the walk is called the length of the walk.

Closed walk: A $v - v_k$ walk of a graph G is called closed if $v_0 = v_k$.

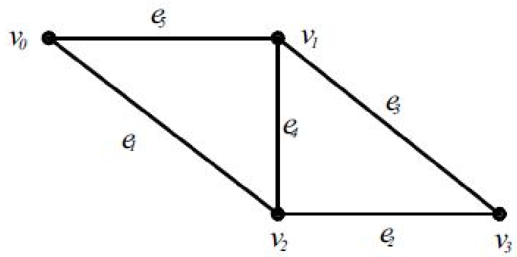
Trail: A walk in which no edge is repeated is called a trail.

Path: A walk in which no vertex is repeated is called a path.

Cycle: A closed walk with n vertices with $n \geq 3$ is called a cycle if all the n-vertices are distinct.

Also note that every path is a trail but every trail is not a path.

For the following graph, $W_1 = v_0e_1v_2e_2v_3e_3v_1$ is a path of length three. $W_2 = v_0e_1v_2e_4v_1e_3v_3$ is a trail as well as path. While $W_3 = v_0e_1v_2e_4v_1e_3v_3e_2v_2$ is a trail but not a path as vertex v_2 is repeated.



A **vertex** u of graph G is said to be **connected** to vertex v of a graph G if there is a path from u to v in graph G.

A graph is called **connected** if every pair of vertices are connected.

A graph which is not connected is called disconnected.

Notation: Given any vertex u of graph G, $C(u)$ denotes the set of all vertices which are connected to u.

The subgraph induced by $C(u)$ is called connected component of a given graph and number of connected components of a graph G is denoted by $\omega(G)$.

Any disconnected graph has at least two connected components. Any connected graph has only one component.

Theorem: A graph G is disconnected if and only if its vertex set V can be partitioned into two non-empty disjoint subsets V_1 and V_2 such that there does not exists any edge in G whose one end vertex is in subset V_1 and other in subset V_2.
Proof:($\Rightarrow$) Suppose that such partition exists. Consider $u, v \in V$ such that $u \in V_1$ and $v \in V_2$. No path exists between u and v; otherwise there would be at least one edge whose one end vertex is in V_1 and the other end vertex is in V_2.

Hence if partition exists then G is disconnected.

($\Leftarrow$) Conversely let G be a disconnected graph and let $u \in V(G)$. Let V_1 be the set of all the vertices joined by a path to u.

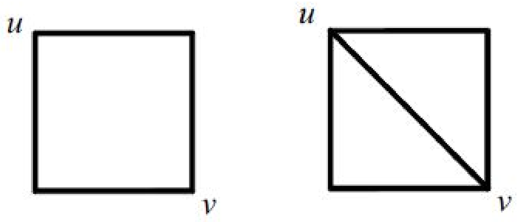
V_1 will not include all vertices of G as G is disconnected. The vertices which does not belongs to V_1 they form the set V_2.

Due to construction of V_1 and V_2 no vertex of V_1 is joined to any vertex of V_2.

Hence V_1 and V_2 together form a partition.

Distance in Graph: If u and v are two vertices of a graph G then the length of shortest path between u and v in G is called the distance between u and v in G and it is denoted as $d(u, v)$.

consider the following graphs:



$d(u,v)=2$ $d(u,v)=1$

If W_1 is a $u - v$ walk and W_2 is a $v - w$ walk. Like $W_1 = ue_1 \ldots e_k v$ and $W_2 = v f_1 \ldots f_i w$ then joining W_1 and W_2, the new walk $W = ue_1 \ldots e_k v f_1 \ldots f_i w$ is a $u - w$ walk. This new walk is called concatenation of two walks.
Theorem: A non-empty graph G with at least two vertices is bipartite if and only if it has no odd cycle.
Proof: Suppose that G is bipartite graph with vertex set V having bipartition $V = V1 \cup V2$. Let $C = v_0, v_1, v_2, \ldots, v_k, v_0$ be a cycle of G. Also assume that $v_0 \in V_1$. Then as G is bipartite $v1 \in V2$ and consequently $v_2 \in V_1$, $v_3 \in V_2$, $v_4 \in V_1$ and so on.

In general the odd index vertices v_{2n+1} must belongs to V_2 while the even indexed vertices v_{2n} must be in V_1. Now since v_0 is in V_1 we must have v_k on other end of cycle must be in V_2. Hence k must be an odd number and consequently the cycle C is even. As C was arbitrary cycle we can say that G has no odd cycles.

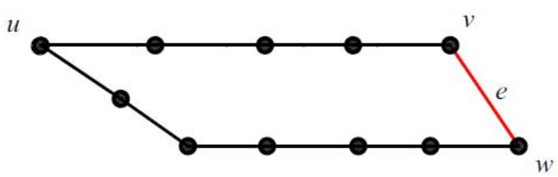
Conversely, let G be a graph with at least two vertices which has no odd cycles. Without loss of generality also assume that G is connected.

For any $u \in V$ define a partition V_1 and V_2 of V as follows:

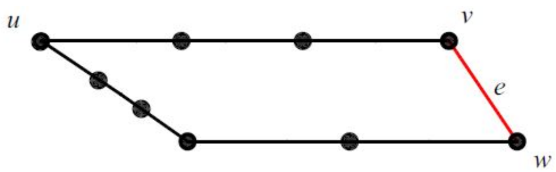
$V_1 = \{x/d(u, x) \text{ is even}\}$, $V_2 = \{\ y/d(u, y) \text{ is odd }\}$

If possible suppose that G is not bipartite. Then there are at least two vertices (say v and w) in either of the sets such that there is an edge e joining them.

If $v, w \in V_1$ then $d(u, w)$ and $d(u, v)$ are even and concatenation of $u - v$ and $u - w$ paths will be even which give rise to an odd cycle together with e as shown in figure.



Similarly if $v, w \in V_2$ then $u - v$ and $u - w$ will be odd and their concatenation will be even which give rise to an odd cycle together with e as shown in the figure.

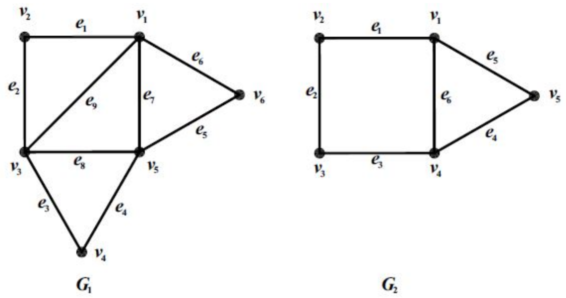


Thus in both the situation G contains an odd cycle which contradicts the fact that G does not have any odd cycle. Therefore V_1 and V_2 form a bipartition of G. Hence G is bipartite.

Eulerian Trail: A trail in G is called an Eulerian trail if it includes every edge of G.

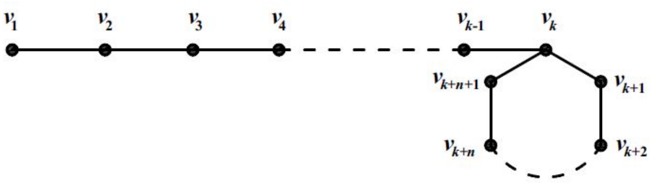
Closed Eulerian Trail: An Eulerian trail with identical end vertices is called a closed Eulerian trail.

Eulerian Graph: A graph is called Euler graph or Eulerian graph if it has a closed Euler trail.
In this figure, G_1 is an Eulerian graph as it has a closed Eulerian trail $v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_6e_6v_1e_7v_5e_8v_3e_9v_1$ while the graph G_2 is not an Eulerian graph but it has an Eulerian trail $v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_1e_6v_4$.
Lemma: Any simple graph G having all the vertices with degree at least two then G contains a



cycle.
Proof: Let v_1 be any vertex of G since $d(v_1) \geq 2$. We can choose an edge e_1 with end vertices v_1 and v_2 say. since $d(v_2) \geq 2$, We can choose an edge e_2 with end vertices v_2 and v_3 say which is different from v_1. Continuing this process in like way at the ith stage, we have an edge e_i with end vertices v_i and v_{i+1} where v_{i+1} is different from any vertex chosen earlier as shown in the figure.

Since G has finitely many vertices, we must have choose a vertex which is chosen earlier(Otherwise vertex of a last edge will have degree 1). If v_k is the first such vertex then first two occurrences of v_k form a cycle as shown in following figure.



Theorem: The following are equivalent for a connected graph G
(1) G is Eulerian.
(2) Every vertex of G has even degree.
(3) The set of edges of G can be partitioned into cycles.

Proof: (1) $\Rightarrow$ (2) As G is an Eulerian graph it will have a closed Eulerian trail C starting and ending at u. If v is a vertex of G other than u then v must be a vertex on the trail C since G being connected and C is an Euler trail. Moreover v is met on the trail C, it is entered and left by different edges (because C is a trail). Thus each occurrence of v in C contributes 2 to $d(v)$. Hence $d(v)$ is even. Now C begins and ends with u, the first and last edges of C contribute 2 to the degree of u and any internal occurrence of u on C will also contribute 2 to the degree of u. Hence $d(u)$ is also even. Thus degree of every vertex of G is even.

(2) $\Rightarrow$ (3)
Since G is connected, nontrivial and every vertex has degree at least 2 then by Lemma-1 G contains a cycle, say C. The removal of an edge of C results in a spanning subgraph G_1 in which every vertex has even degree. If G_1 has no edges then all the edges of G form a cycle and (3) holds. Otherwise the repetition of the argument applied to G_1 results in a graph G_2 in which all the vertices are of even degree. If G_2 has no edges then all the edges of G_1 form a cycle and (3) holds. Otherwise the argument can be repeated until we obtain a totally disconnected graph G_n. Thus n-cycles $G_1, G_2, \ldots, G_n$ form a partition.

(3) $\Rightarrow$ (1)
Let G can partitioned into cycles and C_1 be one such cycle. If C_1 is the only cycle the obviously it contains all the edges of G and hence the result. Otherwise there is another cycle C_2 with vertex v_1 common in C_1. The walk beginning at v_1 and consisting of the cycles C_1 and C_2 in succession is closed trail consisting of edges of these two cycles. By repeating the process, we can construct a closed trail consisting all the edges of G. Hence G is an Eulerian graph.

Theorem: A connected graph G has an Euler trail iff it has either no vertices of odd degree or exactly two vertices of odd degree.
Proof: Suppose G has an Euler trail and v is any vertex other than origin and terminus of the trail. Then $d(v)$ must be even. Thus the only possible odd vertices are the origin and terminus vertices of the trail. Thus the number of odd vertices are exactly two(It can not be one because we have proved that for any graph number of odd vertices are even).

Conversely suppose that u and v be the only vertices of odd degree of a graph G. Add an edge joining u and v then e will increase degree of u and v by one. Now u and v are the vertices of even degree. Consequently $G + e$ will become a connected graph where all the vertices are of even degree. $G + e$ is an Eulerian graph which has a closed Eulerian trail $v_0 e_1 v_1 e_2 \ldots e_n v_n$. We may suppose that $e_1 = e$, $v_0 = u$, $v_1 = v$ and $v_n = u$. Then deleting edge e from above trail gives trail $v_1 e_2 \ldots e_n v_n$ from vertex v to vertex u which involves each edge of G exactly once. Hence graph G has an Eulerian trail which completes the proof.

Note: Some authors define Euler trail by a closed trail. They call an open Euler trail as "Unicursal line". In our accepted terminology Euler trail need not be closed. Therefore for us unicursal line is same as Euler trail. Using above theorem we have an immediate definition of unicursal graph as follows:

Unicursal Graph:: A connected graph with exactly two vertices of odd degree is called unicursal graph. This concept can be generalized in the following form.

Theorem: In a connected graph with exactly $2m$ vertices, with odd degree, there exist m edge disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.
Proof: Let the odd degree vertices of a given graph G as $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_m$. Add m edges to G between the pairs $(u_1, v_1), (u_2, v_2), \ldots, (u_m, v_m)$ to form a new graph G, whose

every vertex is of even degree.

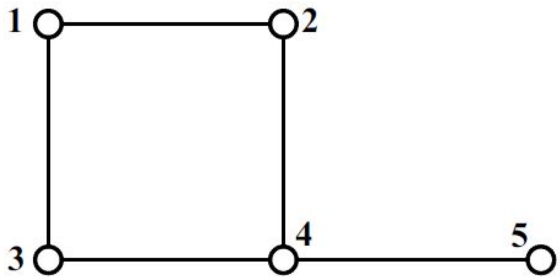
Then G is Euler graph and therefore it consists a closed Euler trail, say C. Now if we remove from C the m edges we just added, C will split into m walks, each of which is a unicursal line. The first removal will result into single unicursal line; the second removal will split that into two unicursal line and each succession will split a unicursal line into two unicursal lines, until there are m of them. And the resultant graph after each removal will have exactly two odd degree vertices. Hence they are m edge disjoint unicursal graphs. Which completes the proof.

Eccentricity Let G be a connected graph with vertex set $V(G)$. For each $v \in V$, the eccentricity of v denoted by $e(v)$ is defined by $e(v) = \max\{d(u,v) : u \in V(G); u \neq v\}$.

Radius: The radius of G, denoted by $rad(G)$ is defined by $rad(G) = \min\{e(v) : v \in V(G)\}$.

Diameter: The diameter of G, denoted by $diamm(G)$ is defined by $diamm(G) = \max\{e(v) : v \in V(G)\} = \max\{d(u,v) : u, v \in V(G)\}$.

Here $e(1) = 3, e(2) = 2, e(3) = 2, e(4) = 2, e(5) = 3$, $rad(G) = 2$ and $diam(G) = 3$. Find



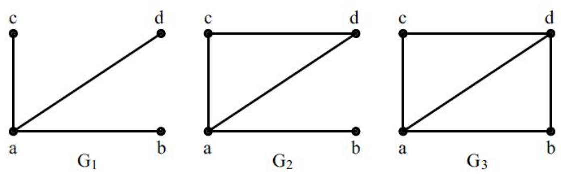
$e(v)$, $v \in V(G)$ where G is Petersen graph, $K_{n,n}, C_n$.

Hamiltonian Path: A Hamiltonian path in a graph G is a path which contains every vertex of G.

Hamiltonian Cycle: A Hamiltonian cycle in a graph G is a cycle which contains every vertex of G.

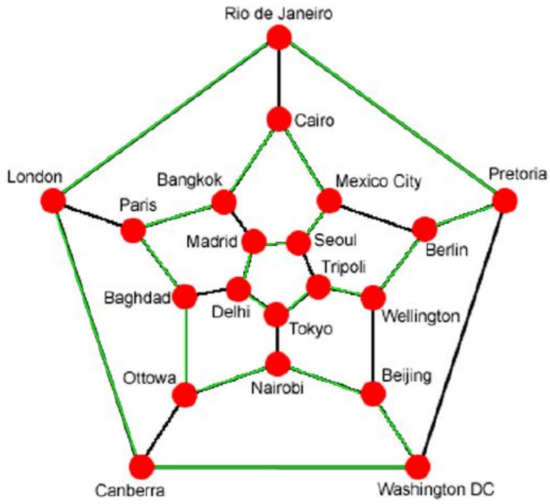
Hamiltonian Graph: A graph G is called Hamiltonian if it has a Hamiltonian cycle.

G_1 has no Hamiltonian path, G_2 has a Hamiltonian path but no Hamiltonian cycle while G_3 has a Hamiltonian cycle acdba. C_n is Hamiltonian graph. K_n is Hamiltonian graph. Super

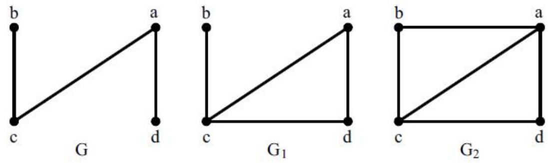


graph of every Hamiltonian graph is Hamiltonian.

Hamiltonian graph are named after Sir William Hamilton, an Irish mathematician(1805 - 1865) who invented a puzzle, called the Icosian game which he sold for 25 guineas to a game manufacturer in Dublin. The puzzle involved a dodecahedron on which each of the 20 vertices was labeled by the name of some capital city in the world. The object of the game to construct using the edge of the dodecahedron, a tour of all the cities which visited each city exactly once beginning and ending at the same city.



Maximal non-Hamiltonian Graph: A simple graph G is called maximal non-Hamiltonian graph if it is not Hamiltonian but addition of an edge between two non-adjacent vertices of G results in a Hamiltonian graph. Here, G_1 is maximal non-hamiltonian graph.

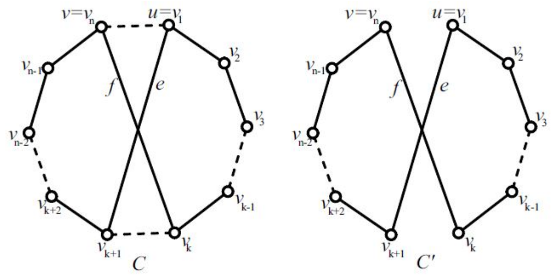


Theorem: If G is a simple graph with n vertices where $n \geq 3$ and $d(v) \geq n/2$ for every vertex v of G is Hamiltonian.
Proof: If possible suppose that the result is not true. Then for some value $n \geq 3$ there is a non Hamiltonian graph in which every vertex has degree at least $n/2$. More over any spanning super graph with the same set of vertices will also have every vertex with degree at least $n/2$ because any proper super graph is obtained by introducing more edges. Thus there will be a maximal non Hamiltonian graph G with n vertices and $d(v) \geq n/2$. Such G is not complete as it is non Hamiltonian. Then there are at least two non adjacent vertices u and v in G. Let $G + uv$ be the super graph of G by adding an edge between u and v. As a result this addition $G + uv$ must be Hamiltonian as G is maximal non Hamiltonian. Then $G+uv$ will contain a Hamiltonian cycle C.

Let $C = v_1v_2 \ldots v_nv_1$ with $v_1 = u$ and $v_n = v$. Now let us consider following two sets. $S = \{v_i \in C$ there is an edge uv_{i+1} in $G\}$ and $T = \{v_j \in C$ there is an edge vv_j in $G\}$ Then $v_n \notin S$ as well as $v_n \notin T$ otherwise the edges uv_1 (interpreting v_{n+1} as v_1) and vv_n are loops which is not possible as G being a simple graph. Thus $v_n \notin S \cup T$. Denoting $|S|$, $|T|$ and $|S \cup T$ as the number of elements in S , T and $S \cup T$ respectively, we get $|S \cup T| < n$ — (1)
Also for every edge incident with u there corresponds precisely one vertex v_i in S thus $|S| = d(u)$ and $|T| = d(v)$ — (2)

More over if vertex $v_k \in S \cup T$ then there is an edge e joining u to v_{k+1} and an edge f joining v to v_k which give rise to cycle C_0 as $C_0 = v_1v_{k+1}v_{k+2} \ldots v_nv_kv_{k+1} \ldots v_2v_1$ as a Hamiltonian cycle in G as shown in figure which is a contradiction as G is non Hamiltonian.

This shows that $S \cap T = \emptyset$ which implies that $|S \cup T| = |S| + |T|$. Hence by (1) and (2) we have $d(u) + d(v) = |S| + |T| = |S \cup T| < n$.

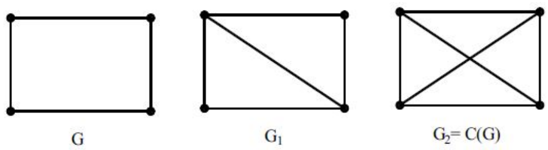


This is not possible since in G, $d(u) \geq n/2$, $d(v) \geq n/2$ and so $d(u)+d(v) \geq n$. This contradiction is due to our wrong assumption. Hence the result holds.

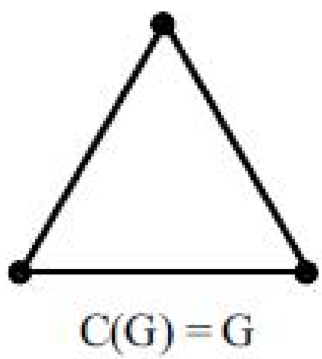
Theorem: If G be a simple graph with n vertices and let u and v be non adjacent vertices in G such that $d(u) + d(v) \geq n$. Let $G + uv$ denote the super graph of G obtained by joining u and v by an edge then G is Hamiltonian iff $G + uv$ is Hamiltonian.
Proof: Suppose that G is Hamiltonian then it will contain a Hamiltonian cycle then its supergraph must contain a Hamiltonian cycle. Therefore $G + uv$ is also Hamiltonian.

Conversely suppose that $G + uv$ is Hamiltonian and if G is non Hamiltonian then applying the procedure adopted in preceding theorem, we reach to the contradiction $d(u) + d(v) < n$ but it is given that $d(u) + d(v) \geq n$. Hence G must be Hamiltonian.

Hamilton Closure: Let G be a simple graph with n vertices. If there are two non-adjacent vertices u_1 and v_1 in G such that $d(u_1) + d(v_1) < n$, join u_1 and v_1 by an edge to form the super graph G_1. Then if there are two nonadjacent vertices u_2 and v_2 such that $d(u_2) + d(v_2) < n$ in G_1 join u_2 and v_2 by an edge to form the super graph G_2. Continuing in this way, recursively joining pairs of non-adjacent vertices where degree sum is at least n until no such pair remains. The final subgraph thus obtained is called the closure of G and is denoted by $C(G)$.



Note: If no two such vertices exist then $C(G) = G$.



The following result is the characterization of Hamiltonian graph depending on closure of a graph.

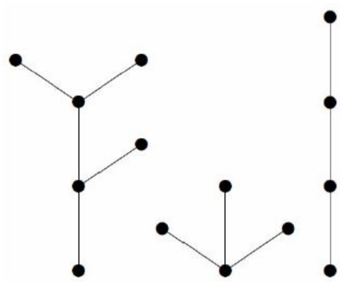
Theorem: A simple graph G is Hamiltonian if and only if its closure $C(G)$ is Hamiltonian.

Proof: Since $C(G)$ being a super graph of G and if G is Hamiltonian then $C(G)$ must be Hamiltonian. Conversely suppose that $C(G)$ is Hamiltonian. Let $G_1, G_2, \ldots, G_{k-1}, G_k = C(G)$ be the sequence of graph obtained by closure operation. Since $C(G) = G_k$ is obtained from G_{k-1} by setting $G_k = G_{k-1} + uv$ where u and v is a pair of non adjacent vertices satisfying $d(u) + d(v) \geq n$ then it follows from the theorem that G_{k-1} is Hamiltonian. Similarly G_{k-2} so $G_{k-3}, \ldots$ so G_1 and so G must be Hamiltonian.

Corollary: Let G be a simple graph on n vertices with $n \geq 3$. If $C(G)$ is complete then G is Hamiltonian.
Proof: It is obvious from the theorem. Since any complete graph is Hamiltonian i.e. If $C(G) = K_n$ then G is Hamiltonian.

Definition: A graph G without cycle is called acyclic graph or forest.

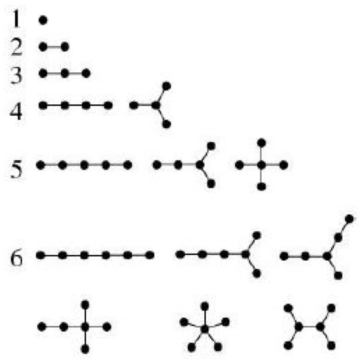
Forest is shown in following figure:



Forest

Definition: An acyclic connected graph is called tree.

In following figure some trees are shown.

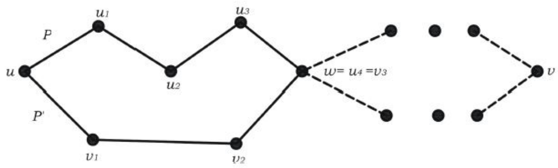


Some trees upto at most 6 vertices.

Theorem:(a) Let u and v be distinct vertices of a tree T. Then there is precisely one path from u to v.

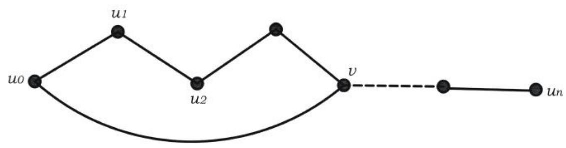
(b) Let G be a simple graph. If for every pair of distinct vertices u and v of G there is a precisely one path from u to v then G is a tree.
Proof: (a) Let us suppose that the result is false. Then there are two different paths from u to v, say $P = uu_1u_2, \ldots, u_mv$ and $P' = uv_1v_2, \ldots, v_nv$. Let w be the first vertex after u which

belongs to both P and P' as shown in following figure. Then $w = u_i = v_j$ for some indices i and j.



This produces the cycle $c = uu1 ::: u_i v_{j-1}, \ldots, v_1 u$. Since T is a tree it has no cycles. This contradiction is due to our wrong assumption. Thus there is precisely one path from u to v.
(b) Since by assumption, there is a path between each pair of vertices u and v implies that G must be connected. Thus to prove the required result it suffices to prove that G has no cycles. If possible G has a cycle $C = v_1 v_2 v_3, \ldots, v_n v_1$, where $n \geq 2$. Since cycle is a trail, the edge $v_n v_1$ does not appear in the path $v_1 v_2, \ldots, v_n$. Thus $P = v_1 v_2, \ldots, v_n$ and $P_0 = v_1 v_n$ are two different paths from v_1 to v_n. This contradicts our assumption of precisely one path between any pair of distinct vertices. Hence G has no cycles. Then $G$is a tree as required.

Theorem: Let T be a tree with at least one edge and $P = u_0 u_1 u_2, \ldots, u_n$ be a path of maximum length in T. Then $d(u_0) = 1 = d(u_n)$.
Proof: If possible let $d(u_0) > 1$ then u_0 is adjacent to u_1 as well as adjacent to some other vertex v of T. If this vertex is one of the vertex form the path P then this situation will give rise to a cycle $C = u_0 u_1 u_2, \ldots, v u_0$ as shown in below Figure.



Which is not possible as T is tree. If u_0 is adjacent to any other vertex which is not on path P. Let w be such vertex. Then $C = w u_0 u_1 u_2, \ldots, u_n$ will produce a path of length $n + 1$ as shown in below Figure. Which is also not possible as it contradicts the maximality of P. Thus $d(u_0) > 1$

is not possible. So $d(u_0) = 1$. By similar argument one can show that $d(u_n) > 1$. Hence the result.

Corollary: Any tree T with at least one edge has more than one vertex of degree 1.
Proof: In such a tree T there is a longest path P. Then according to the result we have proved there are at least two vertices of degree one.

Theorem: Every tree T on n vertices has exactly $n - 1$ edges.
Proof: To prove the result we will use induction on n. When $n = 1$ then T is trivial tree, it has no edges as it has no loops. This proves the result for $n = 1$.

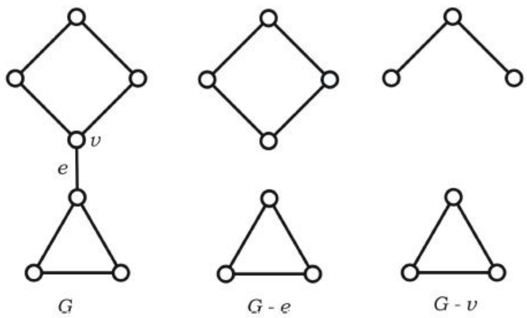
Now suppose that the result is true for $n = k$(where $k \in N$) and we will show that it true for $n = k + 1$. Let T be a tree with $k + 1$ vertices and let u be the vertex of degree 1 in T(such vertex exist due to corollary). Let $e = uv$ be the unique edge of T which has u as an end. If x and y are vertices in T both different form u then any path joining x to y does not go through the vertex u. Otherwise edge e will occur twice in this path. Thus the subgraph $T - u$ obtained form T by deleting the vertex u is connected. Moreover if C is a cycle in $T-u$ then C would be cycle in T is impossible since T is a tree.

Thus the subgraph $T-u$ is acyclic. Hence it is a tree. Since $T-u$ has k vertices and so by induction principle $T-u$ has $k-1$ edges. Since $T-u$ exactly one edge less than T it follows that T has k edges as required. In other words assuming result true for $k \geq N$, we have shown that it is true for $k+1$. Then by mathematical induction it is true for all $k \in N$. Hence the result.

Theorem: Let G be a forest with n vertices and k connected components. Then G has $n-k$ edges.
Proof: Denote the k components of G by $c_1, \ldots, c_k$ and suppose that for each $i, 1 \leq i \leq k$, the ith component c_i has n_i vertices. Then $\sum_{i=1}^{k} n_i = n$. Since each c_i is a tree then it has $n_i - 1$ edges and each edge of G belongs to precisely one component of G and the total number edges in G is $(n_1-1)+(n_2-1)+\ldots+(n_k-1)$. Thus G has total $\sum_{i-1}^{k} n_i - k$ edges i.e. $n-k$ edges.

Definition: An edge e (a vertex v) of a graph G is called a bridge or a cut edge (cut vertex) if the edge (vertex) deleted sub graph $G-e$ $(G-v)$ has more connected components than G.

Consider the graph G as shown below where e is bridge and v is a cut vertex.

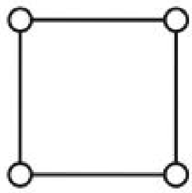


Bridge (cut edge) cut vertex

From the previous Figure one can observe that a bridge is an edge which is the only link between two parts of a graph. Its deletion results into more disjoint parts.

Definition: For a graph $G = (V, E)$ cut set is a connected minimal set of edges whose removal from G renders G disconnected.

Definition: An edge e of a graph G is called a cycle edge if e is a part of some cycle.
In the following Figure e is a cycle edge. In fact all four edges of this graph G are cycle edge.



Theorem: Any edge of a graph G is a bridge if and only if it is not a cycle edge.
Proof: Let e be an edge between vertices u and v of graph G. If e is not a bridge then it is either a loop or there is a path $P = uu_1 \ldots u_n v$ from u to v different from the edge e. If it is a loop then it forms cycle with itself. If there is a path P its concatenation with edge e is $P = uu_1 \ldots u_n vu$ is a cycle in G and e is cycle edge. Thus e is not a bridge then it is cycle edge which is equivalent

to saying that if e is not part of any cycle then e must be a bridge. Conversely suppose that e is a part of some cycle $C = u_0u_1 \ldots u_m$ in G. Let $e = u_iu_{i+1}$ when $m = 1$, $C = u_0u_1$ is a loop. If $m > 1$ then $P = u_iu_{i-1} \ldots u_0u_{m-1} \ldots u_{i+1}$ is a path from u to v different from e which is shown in Figure. Thus e is not a bridge. This shows that if e is a bridge then e is not a part of any cycle.

Theorem: A connected graph G is a tree if and only if every edge of G is a bridge.
Proof: Suppose that G is a tree. Then G is acyclic i.e. no edge of G is a cycle edge. Then if e is any edge of G it is a bridge by the theorem.

Conversely suppose that G is connected and every edge e of G is a bridge. Then G cannot have any cycle since any cycle edge is not a bridge by above theorem. Hence G is acyclic connected graph. so it is a tree.

Theorem: Let G be a graph with n-vertices. Then the following are equivalent.
(i) G is a tree.
(ii) G is acyclic graph with $n-1$ edges.
(iii) G is a connected graph with $n-1$ edges.
Proof: We will prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii)
Suppose G is a tree. Then by its definition G is an acyclic graph and it has $n-1$ edges according to theorem. Thus (ii) holds.

(ii) $\Rightarrow$ (iii)
Suppose that G is acyclic graph with $n-1$ edges and $W(G)$ denoted the number of connected components of G. Then by theorem G has $n - W(G)$ edges. Here $W(G) = 1$ implies that G has only one component i.e. G is connected. Thus (iii) holds.
(iii) $\Rightarrow$ (i)
Suppose that G is connected graph with $n-1$ edge. To prove (i) we have to show that G is acyclic. If possible let G is not acyclic. Then G contains a cycle and any edge of this cycle can not be a bridge according to theorem. Let e be such edges then since e is not a bridge $G-e$ will still remain connected. However $G-e$ has $n-2$ edges and n vertices which is not possible by corollary. This contradiction is due to our assumption that G is not acyclic. Hence G must be acyclic. Thus G is acyclic connected graph with $n-1$ edges. So G is a tree.

Spanning Trees: A spanning tree of a graph G is a spanning subgraph of G that is tree. In following Figure graph G and its spanning tree is shown.