MODULE I

BASICS OF HEAT TRANSFER

1.1 Difference between heat and temperature

In describing heat transfer problems, we often make the mistake of interchangeably using the terms heat and temperature. Actually, there is a distinct difference between the two. *Temperature* is a measure of the amount of energy possessed by the molecules of a substance. It is a relative measure of how hot or cold a substance is and can be used to predict the direction of heat transfer. The usual symbol for temperature is *T*. The scales for measuring temperature in SI units are the Celsius and Kelvin temperature scales. On the other hand, *heat* is energy in transit. The transfer of energy as heat occurs at the molecular level as a result of a temperature difference. The usual symbol for heat is Q. Common units for measuring heat are the Joule and calorie in the SI system.

What is Heat Transfer?
"*Energy in transit due to temperature difference*."

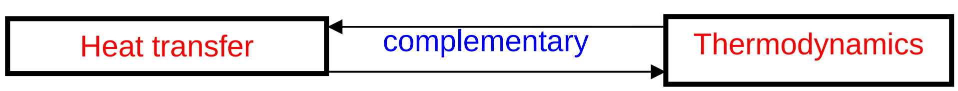
1.2 Difference between thermodynamics and heat transfer

Thermodynamics tells us:

- how much heat is transferred (δQ)
- how much work is done (δW)
- final state of the system

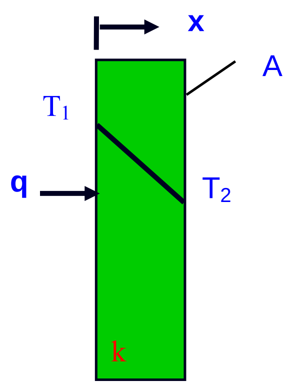
Heat transfer tells us:

- how (with what **modes**) δQ is transferred
- at what **rate** δQ is transferred
- temperature distribution inside the body



1.3 Modes of Heat Transfer

- **Conduction:** An energy transfer across a system boundary due to a temperature difference <u>by the mechanism of inter-molecular interactions</u>. Conduction needs matter and does not require any bulk motion of matter.



Conduction rate equation is described by the Fourier Law:

$$\vec{q} = -kA\nabla T$$

where: q = heat flow vector, (W)
k = thermal conductivity, a thermodynamic property of the material. (W/m K)
A = Cross sectional area in direction of heat flow. (m^2)
∇T = Gradient of temperature (K/m)
$= \partial T/\partial x\ \mathbf{i} + \partial T/\partial y\ \mathbf{j} + \partial T/\partial z\ \mathbf{k}$

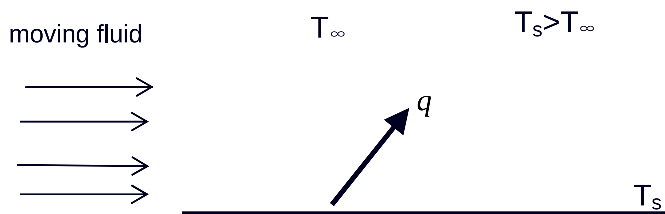
Note: Since this is a vector equation, it is often convenient to work with one component of the vector. For example, in the x direction:

$$q_x = - k\ A_x\ dT/dx$$

In circular coordinates it may convenient to work in the radial direction:

$$q_r = - k\ A_r\ dT/dr$$

- **Convection:** An energy transfer across a system boundary due to a temperature difference <u>by the combined mechanisms of intermolecular interactions and bulk transport.</u> Convection needs fluid matter.



Newton's Law of Cooling:

$$q = h\ A_s\ \Delta T$$

where: q = heat flow from surface, a scalar, (W)
h = heat transfer coefficient (which is not a thermodynamic property of the material, but may depend on geometry of surface, flow characteristics, thermodynamic properties of the fluid, etc. (W/m^2 K)
A_s = Surface area from which convection is occurring. (m^2)
$\Delta T = T_S - T_\infty$ = Temperature Difference between surface and coolant. (K)

Convection → Free or natural convection (induced by buoyancy forces)

Convection → Forced convection (induced by external means)

May occur with phase change (boiling, condensation)

Table 1. Typical values of h (W/m^2K)

Free convection	gases: 2 - 25 liquid: 50 – 100
Forced convection	gases: 25 - 250 liquid: 50 - 20,000
Boiling/Condensation	2500 -100,000

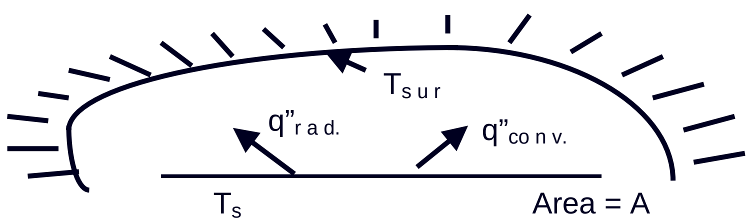
- **Radiation:** Radiation heat transfer involves the transfer of heat by <u>electromagnetic radiation that arises due to the temperature of the body.</u> Radiation does not need matter.

Emissive power of a surface:

$$E=\sigma\varepsilon T_s^4 \ (W/\ m^2)$$

where: ε = emissivity, which is a surface property (ε = 1 is black body)
σ = Steffan Boltzman constant = $5.67 \times 10^{-8}\ W/m^2\ K^4$.
T_s = Absolute temperature of the surface (K)

The above equation is derived from Stefan Boltzman law, which describes a gross heat emission rather than heat transfer. The expression for the actual radiation heat transfer rate between surfaces having arbitrary orientations can be quite complex, and will be dealt with in Module 9. However, the rate of radiation heat exchange between a small surface and a large surrounding is given by the following expression:



$$q = \varepsilon\cdot\sigma\cdot A\cdot(T_s^4 - T_{sur}^4)$$

where: ε = Surface Emissivity
A= Surface Area
T_s = Absolute temperature of surface. (K)
T_{sur} = Absolute temperature of surroundings.(K)

1.4 Thermal Conductivity, k

As noted previously, thermal conductivity is a thermodynamic property of a material. From the State Postulate given in thermodynamics, it may be recalled that thermodynamic properties of pure substances are functions of two independent thermodynamic intensive properties, say temperature and pressure. Thermal conductivity of real gases is largely independent of pressure and may be considered a function of temperature alone. For solids and liquids, properties are largely independent of pressure and depend on temperature alone.

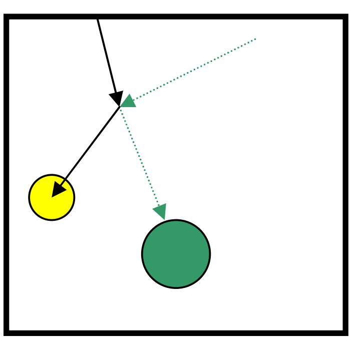
$$k = k\,(T)$$

Table 2 gives the values of thermal conductivity for a variety of materials.

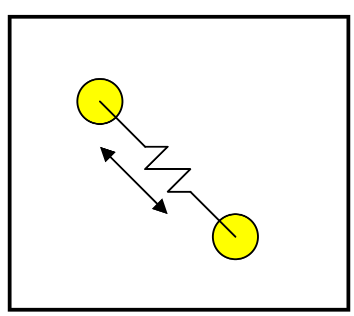
Table 2. Thermal Conductivities of Selected Materials at Room Temperature.

Material	Thermal Conductivity, W/m K
Copper	401
Silver	429
Gold	317
Aluminum	237
Steel	60.5
Limestone	2.15
Bakelite	1.4
Water	0.613
Air	0.0263

It is important that the student gain a basic perspective of the magnitude of thermal conductivity for various materials. The background for this comes from the introductory Chemistry courses. Molecules of various materials gain energy through various mechanisms. Gases exhibit energy through the kinetic energy of the molecule. Energy is gained or lost through collusions of gaseous molecules as they travel through the medium.



Kinetic energy transfer between gaseous molecules.



Lattice vibration may be transferred between molecules as nuclei attract/repel each other.

Solids, being are much more stationary, cannot effectively transfer energy through these same mechanisms. Instead, solids may exhibit energy through vibration or rotation of the nucleus.

Another important mechanism in which materials maintain energy is by shifting electrons into higher orbital rings. In the case of electrical conductors the electrons are weakly bonded to the molecule and can drift from one molecule to another transporting their energy with them. This is an especially effective transport mechanism, so that materials which are excellent electrical conductors are excellent thermal conductors.

Module 1: Short questions

1. What is the driving force for (a) heat transfer (b) electric current flow and (c) fluid flow?

2. Which one of the following is not a property of the material ?
    A. thermal conductivity
    B. heat transfer coefficient
    C. emissivity

3. What is the order of magnitude of thermal conductivity for (a) metals (b) solid insulating materials (c) liquids (d) gases?

4. What is the order of magnitude for the convection heat transfer coefficient in free convection? Forced convection? Boiling?

5. When may one expect radiation heat transfer to be important?

6. An ideal gas is heated from 50 ºC to 70 ºC (a) at constant volume and (b) at constant pressure. For which case do you think the energy required will be greater? Why?

7. A person claims that heat cannot be transferred in a vacuum. How do you respond to this claim?

8. Discuss the mechanism of thermal conduction in gases, liquids and solids.

9. Name some good conductors of heat; some poor conductors.

10. Show that heat flow lines must be normal to isotherms in conduction heat transfer. Will it be true for convection heat transfer?

Module 2: Learning objectives

- The primary purpose of this chapter is to improve your understanding of the conduction rate equation (Fourier's law) and to familiarize you with heat equation. You should know the origin and implication of Fourier's law, and you should understand the key thermal properties and how they vary for different substances. You should also know the physical meaning of each term appearing in the heat equation.
- The student should understand to what form does the heat equation reduce for simplified conditions, and what kinds of boundary conditions may be used for its solution?
- The student should learn to evaluate the heat flow through a 1-D, SS system with no heat sources for rectangular and cylindrical geometries. Many other geometries exist in nature or in common engineering designs. The student, using a similar development, should be able to develop an appropriate equation to describe systems of arbitrary, simple geometry.
- The student should be comfortable with the use of equivalent thermal circuits and with the expressions for the conduction resistances that pertain to each of the three common geometric.
- Composite thermal resistances for 1-D, Steady state heat transfer with no heat sources placed in parallel or in series may be evaluated in a manner similar to electrical resistances placed in parallel or in series.
- The student should learn to evaluate the heat flow through a 1-D, SS system with no heat sources for rectangular and cylindrical geometries.
- In short, by the end of the module, the student should have a fundamental understanding of the conduction process and its mathematical description.

MODULE 2

ONE DIMENSIONAL STEADY STATE HEAT CONDUCTION

2.1 Objectives of conduction analysis:

The primary objective is to determine the temperature field, T(x,y,z,t), in a body (i.e. how temperature varies with position within the body)
T(x,y,z,t) depends on:

- Boundary conditions
- Initial condition
- Material properties (k, cp, ρ)
- Geometry of the body (shape, size)

Why we need T (x, y, z, t)?

- To compute heat flux at any location (using Fourier's eqn.)
- Compute thermal stresses, expansion, deflection due to temp. Etc.
- Design insulation thickness
- Chip temperature calculation
- Heat treatment of metals

2.2 General Conduction Equation

Recognize that heat transfer involves an energy transfer across a system boundary. A logical place to begin studying such process is from Conservation of Energy (1st Law of Thermodynamics) for a closed system:

$$\left.\frac{dE}{dt}\right|_{system} = \dot{Q}_{in} - \dot{W}_{out}$$

The sign convention on work is such that negative work out is positive work in.

$$\left.\frac{dE}{dt}\right|_{system} = \dot{Q}_{in} + \dot{W}_{in}$$

The work in term could describe an electric current flow across the system boundary and through a resistance inside the system. Alternatively it could describe a shaft turning across the system boundary and overcoming friction within the system. The net effect in either case would cause the internal energy of the system to rise. In heat transfer we generalize all such terms as "heat sources".

$$\left.\frac{dE}{dt}\right|_{system} = \dot{Q}_{in} + \dot{Q}_{gen}$$

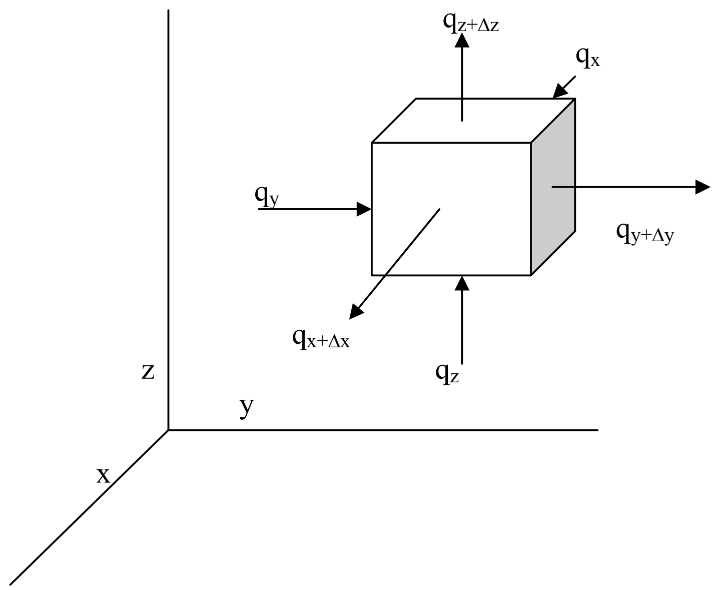
The energy of the system will in general include internal energy, U, potential energy, ½ mgz, or kinetic energy, ½ mΥ^2. In case of heat transfer problems, the latter two terms could often be neglected. In this case,

$$E = U = m \cdot u = m \cdot c_p \cdot \left(T - T_{ref}\right) = \rho \cdot V \cdot c_p \cdot \left(T - T_{ref}\right)$$

where T_{ref} is the reference temperature at which the energy of the system is defined as zero. When we differentiate the above expression with respect to time, the reference temperature, being constant, disappears:

$$\rho \cdot c_p \cdot V \cdot \left.\frac{dT}{dt}\right|_{system} = \dot{Q}_{in} + \dot{Q}_{gen}$$

Consider the differential control element shown below. Heat is assumed to flow through the element in the positive directions as shown by the 6 heat vectors.



In the equation above we substitute the 6 heat inflows/outflows using the appropriate sign:

$$\rho \cdot c_p \cdot (\Delta x \cdot \Delta y \cdot \Delta z) \cdot \left.\frac{dT}{dt}\right|_{system} = q_x - q_{x+\Delta x} + q_y - q_{y+\Delta y} + q_z - q_{z+\Delta z} + \dot{Q}_{gen}$$

Substitute for each of the conduction terms using the Fourier Law:

$$\begin{aligned}\rho \cdot c_p \cdot (\Delta x \cdot \Delta y \cdot \Delta z) \cdot \left.\frac{\partial T}{\partial t}\right|_{system} &= \left\{ -k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} - \left[-k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} \right) \cdot \Delta x \right] \right\} \\ &+ \left\{ -k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} - \left[-k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left(-k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} \right) \cdot \Delta y \right] \right\} \\ &+ \left\{ -k \cdot (\Delta x \cdot \Delta y) \cdot \frac{\partial T}{\partial z} + \left[-k \cdot (\Delta x \cdot \Delta y) \cdot \frac{\partial T}{\partial z} + \frac{\partial}{\partial z}\left(-k \cdot (\Delta x \cdot \Delta y) \cdot \frac{\partial T}{\partial z} \right) \cdot \Delta z \right] \right\} \\ &+ \ddot{q} \cdot (\Delta x \cdot \Delta y \cdot \Delta z)\end{aligned}$$

where $\ddot{q}$ is defined as the internal heat generation per unit volume.

The above equation reduces to:

$$\rho \cdot c_p \cdot (\Delta x \cdot \Delta y \cdot \Delta z) \cdot \left.\frac{dT}{dt}\right|_{system} = \left\{ -\left[\frac{\partial}{\partial x}\left(-k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} \right) \right] \cdot \Delta x \right\}$$

$$+\left\{-\left[\frac{\partial}{\partial y}\left(-k\cdot(\Delta x\cdot\Delta z)\cdot\frac{\partial T}{\partial y}\right)\cdot\Delta y\right]\right\}$$

$$+\left\{\left[\frac{\partial}{\partial z}\left(-k\cdot\left(\Delta x\cdot\Delta y\right)\cdot\frac{\partial T}{\partial z}\right)\cdot\Delta z\right]\right\}+\ddot{q}\cdot\left(\Delta x\cdot\Delta y\cdot\Delta z\right)$$

Dividing by the volume (Δx·Δy·Δz),

$$\rho\cdot c_p\cdot\left.\frac{dT}{dt}\right|_{system}=-\frac{\partial}{\partial x}\left(-k\cdot\frac{\partial T}{\partial x}\right)-\frac{\partial}{\partial y}\left(-k\cdot\frac{\partial T}{\partial y}\right)-\frac{\partial}{\partial z}\left(-k\cdot\frac{\partial T}{\partial z}\right)+\ddot{q}$$

which is the **general conduction equation** in three dimensions.

In the case where k is independent of x, y and z then

$$\frac{\rho\cdot c_p}{k}\cdot\left.\frac{dT}{dt}\right|_{system}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\ddot{q}}{k}$$

Define the thermodynamic property, α, the thermal diffusivity:

$$\alpha\equiv\frac{k}{\rho\cdot c_p}$$

Then

$$\frac{1}{\alpha}\cdot\left.\frac{dT}{dt}\right|_{system}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\ddot{q}}{k}$$

or, :

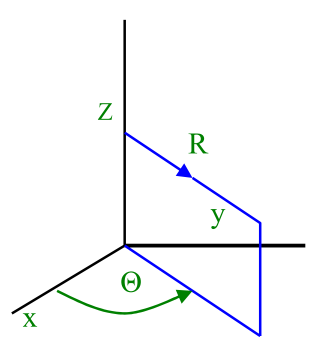
$$\frac{1}{\alpha}\cdot\left.\frac{dT}{dt}\right|_{system}=\nabla^2 T+\frac{\ddot{q}}{k}$$

The vector form of this equation is quite compact and is the most general form. However, we often find it convenient to expand the del-squared term in specific coordinate systems:

Cartesian Coordinates

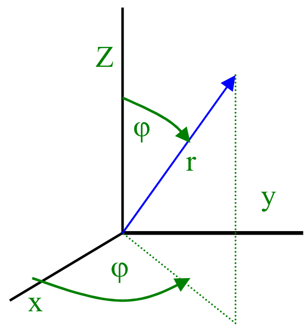
$$\frac{1}{a}\cdot\frac{\partial T}{\partial\tau}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dddot{q}}{k}$$

Circular Coordinates



$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\cdot\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dddot{q}}{k}$$

Spherical Coordinates



$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\cdot\sin^2\theta}\cdot\frac{\partial^2 T}{\partial \phi^2}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial \theta}\left(\sin\theta\cdot\frac{\partial T}{\partial z}\right)+\frac{\dddot{q}}{k}$$

In each equation the dependent variable, T, is a function of 4 independent variables, (x,y,z,τ); (r,θ,z,τ); (r,φ,θ,τ) and is a 2nd order, partial differential equation. The solution of such equations will normally require a numerical solution. For the present, we shall simply look at the simplifications that can be made to the equations to describe specific problems.

- Steady State: Steady state solutions imply that the system conditions are not changing with time. Thus $\partial T/\partial\tau = 0$.
- One dimensional: If heat is flowing in only one coordinate direction, then it follows that there is no temperature gradient in the other two directions. Thus the two partials associated with these directions are equal to zero.
- Two dimensional: If heat is flowing in only two coordinate directions, then it follows that there is no temperature gradient in the third direction. Thus the partial derivative associated with this third direction is equal to zero.
- No Sources: If there are no heat sources within the system then the term, $\dddot{q} = 0$.

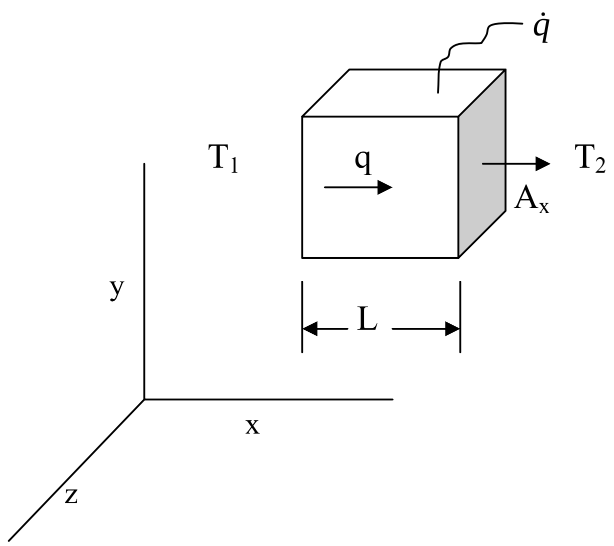
Note that the equation is 2nd order in each coordinate direction so that integration will result in 2 constants of integration. To evaluate these constants two additional equations must be written for each coordinate direction based on the physical conditions of the problem. Such equations are termed “boundary conditions’.

2.3 Boundary and Initial Conditions

- The objective of deriving the heat diffusion equation is to determine the temperature distribution within the conducting body.

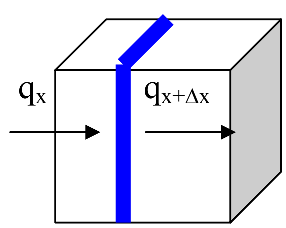
- We have set up a differential equation, with T as the dependent variable. The solution will give us T(x,y,z). Solution depends on boundary conditions (BC) and initial conditions (IC).
- How many BC's and IC's ?
  - Heat equation is second order in spatial coordinate. Hence, 2 BC's needed for each coordinate.
    - 1D problem: 2 BC in x-direction
    - 2D problem: 2 BC in x-direction, 2 in y-direction
    - 3D problem: 2 in x-dir., 2 in y-dir., and 2 in z-dir.
  - Heat equation is first order in time. Hence one IC needed.

2.4 Heat Diffusion Equation for a One Dimensional System



Consider the system shown above. The top, bottom, front and back of the cube are insulated, so that heat can be conducted through the cube only in the x direction. The internal heat generation per unit volume is $\dot{q}$ (W/m^3).

Consider the heat flow through an arbitrary differential element of the cube.



From the 1st Law we write for the element:

$$(\dot{E}_{in} - \dot{E}_{out}) + \dot{E}_{gen} = \dot{E}_{st} \tag{2.1}$$

$$q_x - q_{x+\Delta x} + A_x(\Delta x)\dot{q} = \frac{\partial E}{\partial t} \tag{2.2}$$

$$q_x = -kA_x \frac{\partial T}{\partial x} \tag{2.3}$$

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x}\Delta x \tag{2.4}$$

$$-kA\frac{\partial T}{\partial x} + kA\frac{\partial T}{\partial x} + A\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\Delta x + A\Delta x\,\dot{q} = \rho\ Ac\ \Delta x\frac{\partial T}{\partial t} \tag{2.5}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = \rho\, c\,\Delta x\frac{\partial T}{\partial t} \tag{2.6}$$

Longitudinal conduction

Internal heat generation

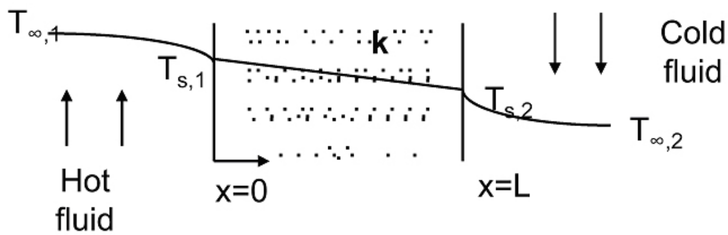
Thermal inertia

If k is a constant, then $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\dot{q}}{k} = \dfrac{\rho\, c}{k}\dfrac{\partial T}{\partial t} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$ (2.7)

- For T to rise, LHS must be positive (heat input is positive)
- For a fixed heat input, T rises faster for higher α
- In this special case, heat flow is 1D. If sides were not insulated, heat flow could be 2D, 3D.

2.5 One Dimensional Steady State Heat Conduction

The plane wall:



The differential equation governing heat diffusion is: $\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$

With constant k, the above equation may be integrated twice to obtain the general solution:

$$T(x)=C_1x+C_2$$

where C_1 and C_2 are constants of integration. To obtain the constants of integration, we apply the boundary conditions at $x = 0$ and $x = L$, in which case

$$T(0)=T_{s,1} \qquad \text{and} \qquad T(L)=T_{s,2}$$

Once the constants of integration are substituted into the general equation, the temperature distribution is obtained:

$$T(x)=(T_{s,2}-T_{s,1})\frac{x}{L}+T_{s,1}$$

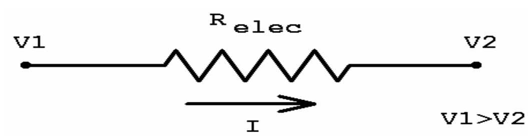
The heat flow rate across the wall is given by:

$$q_x=-kA\frac{dT}{dx}=\frac{kA}{L}\left(T_{s,1}-T_{s,2}\right)=\frac{T_{s,1}-T_{s,2}}{L/kA}$$

Thermal resistance (electrical analogy):

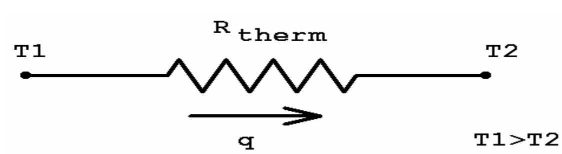
Physical systems are said to be analogous if that obey the same mathematical equation. The above relations can be put into the form of Ohm's law:

V=IR$_{\mathbf{elec}}$



Using this terminology it is common to speak of a thermal resistance:

$$\Delta T = qR_{therm}$$



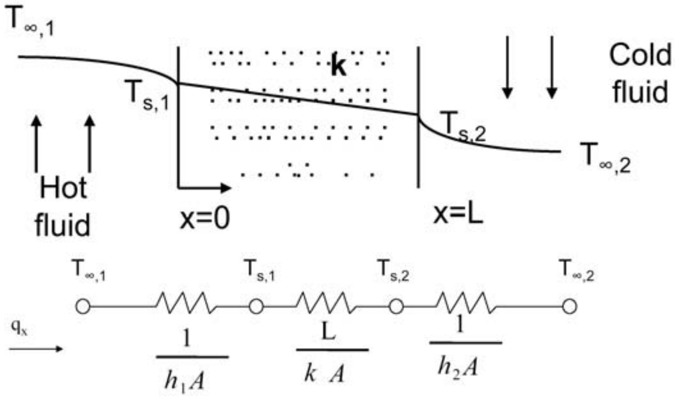
A thermal resistance may also be associated with heat transfer by convection at a surface. From Newton's law of cooling,

$$q=hA(T_s-T_\infty)$$

the thermal resistance for convection is then

$$R_{t,conv}=\frac{T_s-T_\infty}{q}=\frac{1}{hA}$$

Applying thermal resistance concept to the plane wall, the equivalent thermal circuit for the plane wall with convection boundary conditions is shown in the figure below



The heat transfer rate may be determined from separate consideration of each element in the network. Since q_x is constant throughout the network, it follows that

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

In terms of the overall temperature difference $T_{\infty,1} - T_{\infty,2}$, and the total thermal resistance R_{tot}, the heat transfer rate may also be expressed as

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}}$$

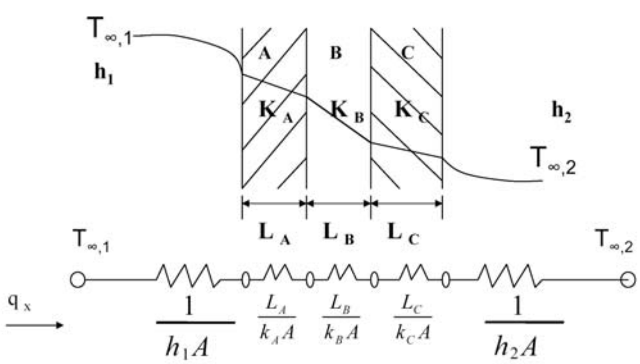
Since the resistance are in series, it follows that

$$R_{tot} = \sum R_t = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

Composite walls:

Thermal Resistances in Series:

Consider three blocks, A, B and C, as shown. They are insulated on top, bottom, front and back. Since the energy will flow first through block A and then through blocks B and C, we say that these blocks are thermally in a series arrangement.



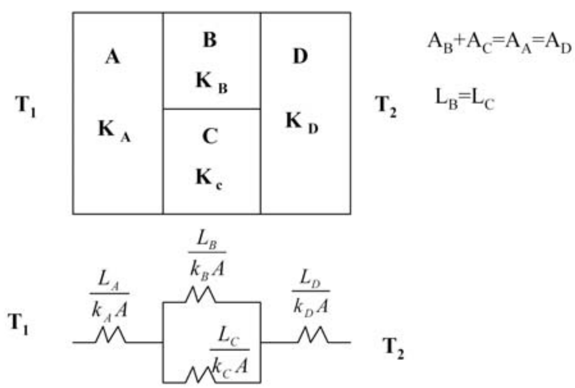
The steady state heat flow rate through the walls is given by:

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{\sum R_t} = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{h_1 A} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2 A}} = UA\Delta T$$

where $U = \frac{1}{R_{tot} A}$ is the overall heat transfer coefficient. In the above case, U is expressed as

$$U = \frac{1}{\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2}}$$

Series-parallel arrangement:

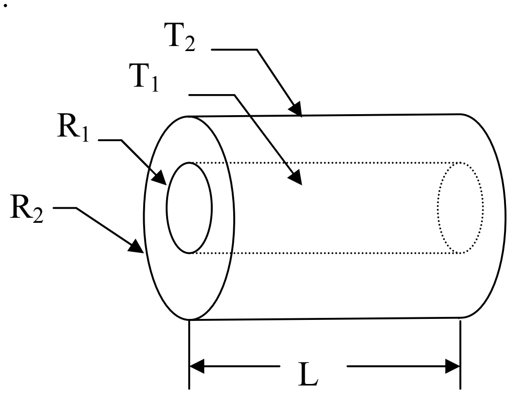


The following assumptions are made with regard to the above thermal resistance model:

1) Face between B and C is insulated.
2) Uniform temperature at any face normal to X.

1-D radial conduction through a cylinder:

One frequently encountered problem is that of heat flow through the walls of a pipe or through the insulation placed around a pipe. Consider the cylinder shown. The pipe is either insulated on the ends or is of sufficient length, L, that heat losses through the ends is negligible. Assume no heat sources within the wall of the tube. If $T_1>T_2$, heat will flow outward, radially, from the inside radius, R_1, to the outside radius, R_2. The process will be described by the Fourier Law.



The differential equation governing heat diffusion is: $\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$

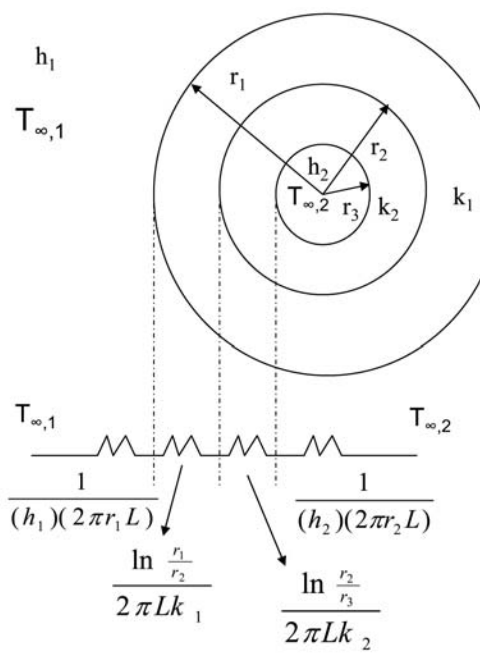
With constant k, the solution is

The heat flow rate across the wall is given by:

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) = \frac{T_{s,1} - T_{s,2}}{L/kA}$$

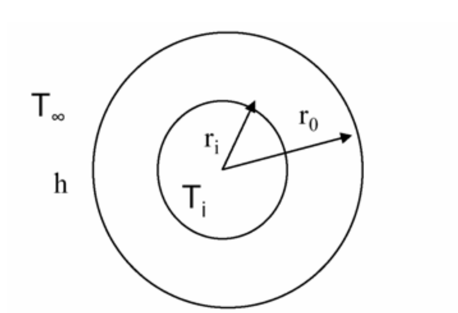
Hence, the thermal resistance in this case can be expressed as: $\frac{\ln \frac{r_1}{r_2}}{2\pi kL}$

Composite cylindrical walls:



$$q_r = \frac{T_{\infty,2} - T_{\infty,1}}{\sum R_t}$$

Critical Insulation Thickness :



$$R_{tot} = \frac{\ln(\frac{r_0}{r_i})}{2\pi kL} + \frac{1}{(2\pi r_0 L)h}$$

Insulation thickness : r_o-r_i

Objective : decrease q , increase R_{tot}

Vary r_o ; as r_o increases, first term increases, second term decreases.

This is a maximum – minimum problem. The point of extrema can be found by setting

$$\frac{dR_{tot}}{dr_0} = 0$$

or,
$$\frac{1}{2\pi k r_0 L} - \frac{1}{2\pi h L r_o^2} = 0$$

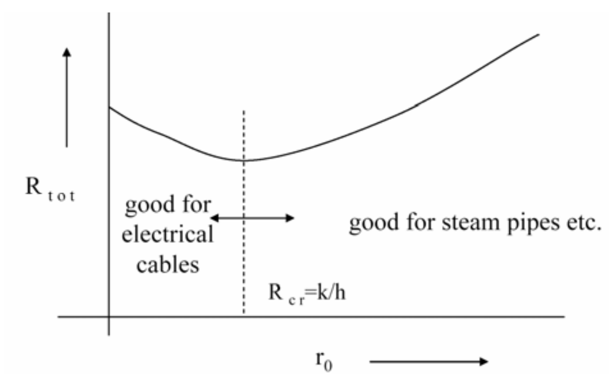
or,
$$r_0 = \frac{k}{h}$$

In order to determine if it is a maxima or a minima, we make the second derivative zero:

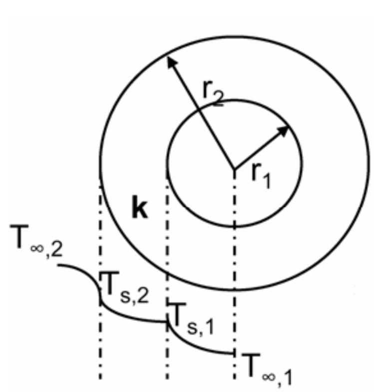
$$\frac{d^2 R_{tot}}{dr_o^2} = 0 \quad \text{at} \quad r_0 = \frac{k}{h}$$

$$\frac{d^2 R_{tot}}{dr_o^2} = \frac{-1}{2\pi k r_o^2 L} + \left.\frac{1}{\pi r_o^2 h L}\right|_{r_0=\frac{k}{h}} \qquad = \frac{h^2}{2\pi L k^3} \rangle 0$$

Minimum q at r_o =(k/h) = r_{cr} (critical radius)



1-D radial conduction in a sphere:



$$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right)=0$$

$$\rightarrow T(r)=T_{s,1}-\{T_{s,1}-T_{s,2}\}\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$$

$$\rightarrow q_r=-kA\frac{dT}{dr}=\frac{4\pi k(T_{s,1}-T_{s,2})}{(1/r_1-1/r_2)}$$

$$\rightarrow R_{t,cond}=\frac{1/r_1-1/r_2}{4\pi k}$$

2.6 Summary of Electrical Analogy

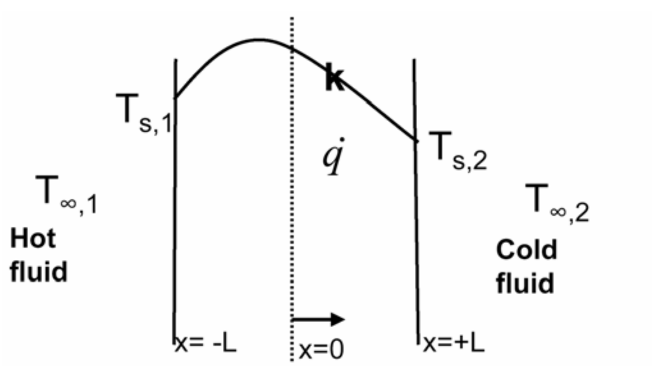
System	Current	Resistance	Potential Difference
Electrical	I	R	ΔV
Cartesian Conduction	q	$\frac{L}{kA}$	ΔT
Cylindrical Conduction	q	$\frac{\ln r_2/r_1}{2\pi kL}$	ΔT
Conduction through sphere	q	$\frac{1/r_1-1/r_2}{4\pi k}$	ΔT
Convection	q	$\frac{1}{h\cdot A_s}$	ΔT

2.7 One-Dimensional Steady State Conduction with Internal Heat Generation

Applications: current carrying conductor, chemically reacting systems, nuclear reactors.

Energy generated per unit volume is given by $\dot{q}=\frac{\dot{E}}{V}$

Plane wall with heat source: Assumptions: 1D, steady state, constant k, uniform $\dot{q}$



$$\frac{d^2T}{dx^2}+\frac{\dot{q}}{k}=0$$

Boundary cond.: $x=-L, \quad T=T_{s,1}$

$x=+L, \quad T=T_{s,2}$

Solution: $T=-\frac{\dot{q}}{2k}x^2+C_1x+C_2$

Use boundary conditions to find C_1 and C_2

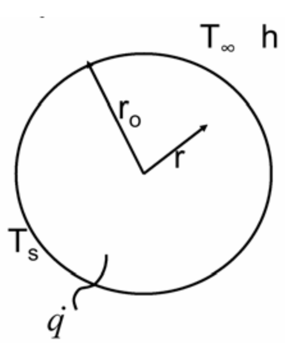
Final solution: $T=\frac{\dot{q}L^2}{2k}\left(1-\frac{x^2}{L^2}\right)+\frac{T_{s,2}-T_{s,1}}{2}\frac{x}{L}+\frac{T_{s,2}+T_{s,1}}{2}$

Heat flux: $q_x''=-k\frac{dT}{dx}$

Note: From the above expressions, it may be observed that the solution for temperature is no longer linear. As an exercise, show that the expression for heat flux is no longer independent of *x. Hence thermal resistance concept is not correct to use when there is internal heat generation.*

<u>Cylinder with heat source:</u> Assumptions: 1D, steady state, constant k, uniform $\dot{q}$

Start with 1D heat equation in cylindrical co-ordinates



$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

Boundary cond.: $r=r_0, \quad T=T_s$

$r=0, \quad \frac{dT}{dr}=0$

Solution: $T(r)=\frac{\dot{q}}{4k}r_0^{\,2}\left(1-\frac{r^2}{r_0^{\,2}}\right)+T_s$

Exercise: T_s may not be known. Instead, T_∞ and h may be specified. Eliminate T_s, using T_∞ and h.

Module 2: Short questions

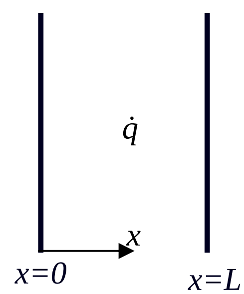
1. How does transient heat transfer differ from steady state heat transfer?

2. What is meant by the term "one-dimensional" when applied to conduction heat transfer?

3. What is meant by thermal resistance? Under what assumptions can the concept of thermal resistance be applied in a straightforward manner?

4. For heat transfer through a single cylindrical shell with convection on the outside, there is a value for the shell radius for a nonzero shell thickness at which the heat flux is maximized. This value is
   (A) k/h
   (B) h/k
   (C) h/r
   (D) r/h

5. The steady temperature profile in a one-dimensional heat transfer across a plane slab of thickness L and with uniform heat generation, $\dot{q}$, has one maximum. If the slab is cooled by convection at $x = 0$ and insulated at $x = L$, the maximum occurs at a value of x given by



   (A) 0
   (B) $\frac{L}{2}$
   (C) $\frac{\dot{q}}{k}$
   (D) L

6. Consider a cold canned drink left on a dinner table. Would you model the heat transfer to the drink as one-, two-, or three-dimensional? Would the heat transfer be steady or transient? Also, which coordinate system would you use to analyse this heat transfer problem, and where would you place the origin?

7. Consider a round potato being baked in an oven? Would you model the heat transfer to the potato as one-, two-, or three-dimensional? Would the heat transfer be steady or transient? Also, which coordinate system would you use to analyse this heat transfer problem, and where would you place the origin?

8. Consider an egg being cooked in boiling water in a pan? Would you model the heat transfer to the egg as one-, two-, or three-dimensional? Would the heat transfer be steady or transient? Also, which coordinate system would you use to analyse this heat transfer problem, and where would you place the origin?

Learning Objectives:

- Students should recognize the fin equation.
- Students should know the 2 general solutions to the fin equation.
- Students should be able to write boundary conditions for (a) very long fins, (b) insulated tip fins, (c) convective tip fins and (d) fins with a specified tip temperature.
- Students should be able to apply the boundary conditions to the fin equation and obtain a temperature profile.
- Students should be able to apply the temperature profile to the Fourier Law to obtain a heat flow through the fin.
- Students should be able to apply the concept of fin efficiency to define an equivalent thermal resistance for a fin.
- Students should be able to incorporate fins into an overall electrical network to solve 1-D, SS problems with no sources.

MODULE 3

Extended Surface Heat Transfer

3.1 Introduction:

Convection: Heat transfer between a solid surface and a moving fluid is governed by the Newton's cooling law: $q = hA(T_s - T_\infty)$, where T_s is the surface temperature and T_∞ is the fluid temperature. Therefore, to increase the convective heat transfer, one can

- Increase the temperature difference $(T_s - T_\infty)$ between the surface and the fluid.
- Increase the convection coefficient h. This can be accomplished by increasing the fluid flow over the surface since h is a function of the flow velocity and the higher the velocity, the higher the h. Example: a cooling fan.
- Increase the contact surface area A. Example: a heat sink with fins.

Many times, when the first option is not in our control and the second option (i.e. increasing h) is already stretched to its limit, we are left with the only alternative of increasing the effective surface area by using fins or extended surfaces. Fins are protrusions from the base surface into the cooling fluid, so that the extra surface of the protrusions is also in contact with the fluid. Most of you have encountered cooling fins on air-cooled engines (motorcycles, portable generators, etc.), electronic equipment (CPUs), automobile radiators, air conditioning equipment (condensers) and elsewhere.

3.2 Extended surface analysis:

In this module, consideration will be limited to steady state analysis of rectangular or pin fins of constant cross sectional area. Annular fins or fins involving a tapered cross section may be analyzed by similar methods, but will involve solution of more complicated equations which result. Numerical methods of integration or computer programs can be used to advantage in such cases.

We start with the General Conduction Equation:

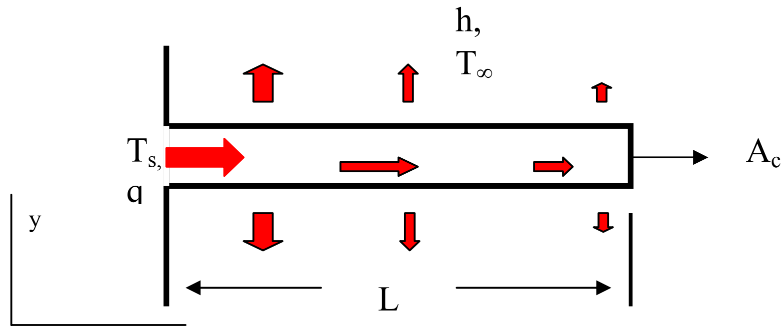
$$\frac{1}{\alpha} \cdot \left.\frac{dT}{d\tau}\right|_{system} = \nabla^2 T + \frac{\ddot{q}}{k} \tag{1}$$

After making the assumptions of Steady State, One-Dimensional Conduction, this equation reduces to the form:

$$\frac{d^2T}{dx^2} + \frac{\ddot{q}}{k} = 0 \tag{2}$$

This is a second order, ordinary differential equation and will require 2 boundary conditions to evaluate the two constants of integration that will arise.

Consider the cooling fin shown below:

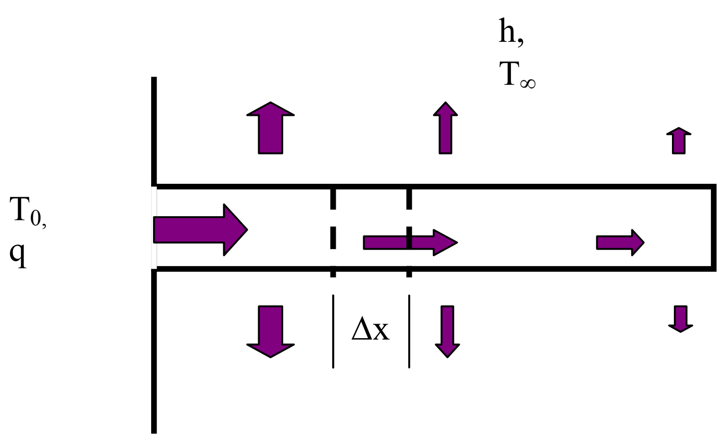


The fin is situated on the surface of a hot surface at T_s and surrounded by a coolant at temperature T_∞, which cools with convective coefficient, h. The fin has a cross sectional area, A_c, (This is the area through with heat is conducted.) and an overall length, L.

Note that as energy is conducted down the length of the fin, some portion is lost, by convection, from the sides. Thus the heat flow varies along the length of the fin.

We further note that the arrows indicating the direction of heat flow point in both the x and y directions. This is an indication that this is truly a two- or three-dimensional heat flow, depending on the geometry of the fin. However, quite often, it is convenient to analyse a fin by examining an equivalent one–dimensional system. The equivalent system will involve the introduction of heat sinks (negative heat sources), which remove an amount of energy equivalent to what would be lost through the sides by convection.

Consider a differential length of the fin.



Across this segment the heat loss will be $h \cdot (P \cdot \Delta x) \cdot (T - T_\infty)$, where P is the perimeter around the fin. The equivalent heat sink would be $\ddot{q} \cdot (A_c \cdot \Delta x)$.

Equating the heat source to the convective loss:

$$\ddot{q} = \frac{-h \cdot P \cdot \left(T - T_\infty\right)}{A_c} \tag{3}$$

Substitute this value into the General Conduction Equation as simplified for One-Dimension, Steady State Conduction with Sources:

$$\frac{d^2T}{dx^2} - \frac{h \cdot P}{k \cdot A_c} \cdot \left(T - T_\infty\right) = 0 \tag{4}$$

which is the equation for a fin with a constant cross sectional area. This is the Second Order Differential Equation that we will solve for each fin analysis. Prior to solving, a couple of simplifications should be noted. First, we see that h, P, k and A_c are all independent of x in the defined system (They may not be constant if a more general analysis is desired.). We replace this ratio with a constant. Let

$$m^2 = \frac{h \cdot P}{k \cdot A_c} \tag{5}$$

then:

$$\frac{d^2T}{dx^2} - m^2 \cdot \left(T - T_\infty\right) = 0 \tag{6}$$

Next we notice that the equation is non-homogeneous (due to the T_∞ term). Recall that non-homogeneous differential equations require both a general and a particular solution. We can make this equation homogeneous by introducing the temperature relative to the surroundings:

$$\theta \equiv T - T_\infty \tag{7}$$

Differentiating this equation we find:

$$\frac{d\theta}{dx} = \frac{dT}{dx} + 0 \tag{8}$$

Differentiate a second time:

$$\frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2} \tag{9}$$

Substitute into the Fin Equation:

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta = 0 \tag{10}$$

This equation is a Second Order, Homogeneous Differential Equation.

3.3 Solution of the Fin Equation

We apply a standard technique for solving a second order homogeneous linear differential equation.

Try $\theta = e^{\alpha \cdot x}$. Differentiate this expression twice:

$$\frac{d\theta}{dx} = \alpha \cdot e^{\alpha \cdot x} \tag{11}$$

$$\frac{d^2\theta}{dx^2} = \alpha^2 \cdot e^{\alpha \cdot x} \tag{12}$$

Substitute this trial solution into the differential equation:

$$\alpha^2 \cdot e^{\alpha \cdot x} - m^2 \cdot e^{\alpha \cdot x} = 0 \tag{13}$$

Equation (13) provides the following relation:

$$\alpha = \pm\ m \tag{14}$$

We now have two solutions to the equation. The general solution to the above differential equation will be a linear combination of each of the independent solutions.

Then:

$$\theta = A \cdot e^{m \cdot x} + B \cdot e^{-m \cdot x} \tag{15}$$

where A and B are arbitrary constants which need to be determined from the boundary conditions. Note that it is a 2nd order differential equation, and hence we need two boundary conditions to determine the two constants of integration.

An alternative solution can be obtained as follows: Note that the hyperbolic sin, sinh, the hyperbolic cosine, cosh, are defined as:

$$\sinh(m \cdot x) = \frac{e^{m \cdot x} - e^{-m \cdot x}}{2} \qquad \cosh(m \cdot x) = \frac{e^{m \cdot x} + e^{-m \cdot x}}{2} \tag{16}$$

We may write:

$$C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) = C \cdot \frac{e^{m \cdot x} + e^{-m \div x}}{2} + D \cdot \frac{e^{m \cdot x} - e^{-m \div x}}{2} = \frac{C + D}{2} \cdot e^{m \cdot x} + \frac{C - D}{2} \cdot e^{-m \cdot x} \tag{17}$$

We see that if (C+D)/2 replaces A and (C-D)/2 replaces B then the two solutions are equivalent.

$$\theta = C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) \tag{18}$$

Generally the exponential solution is used for very long fins, the hyperbolic solutions for other cases.

Boundary Conditions:

Since the solution results in 2 constants of integration we require 2 boundary conditions. The first one is obvious, as one end of the fin will be attached to a hot surface and will come into thermal equilibrium with that surface. Hence, at the fin base,

$$\theta(0) = T_0 - T_\infty \equiv \theta_0 \tag{19}$$

The second boundary condition depends on the condition imposed at the other end of the fin. There are various possibilities, as described below.

Very long fins:
For very long fins, the end located a long distance from the heat source will approach the temperature of the surroundings. Hence,

$$\theta(\infty) = 0 \tag{20}$$

Substitute the second condition into the exponential solution of the fin equation:

$$\theta(\infty) = 0 = A\cdot \overset{\infty}{e^{m\cdot\infty}} + B\cdot \overset{0}{e^{-m\cdot\infty}} \tag{21}$$

The first exponential term is infinite and the second is equal to zero. The only way that this equation can be valid is if A = 0. Now apply the second boundary condition.

$$\theta(0) = \theta_0 = B\cdot e^{-m\cdot 0} \Rightarrow B = \theta_0 \tag{22}$$

The general temperature profile for a very long fin is then:

$$\theta(x) = \theta_0 \cdot e^{-m\cdot x} \tag{23}$$

If we wish to find the heat flow through the fin, we may apply Fourier Law:

$$q = -k\cdot A_c \cdot \frac{dT}{dx} = -k\cdot A_c \cdot \frac{d\theta}{dx} \tag{24}$$

Differentiate the temperature profile:

$$\frac{d\theta}{dx} = -\theta_o \cdot m\cdot e^{-m\cdot x} \tag{25}$$

So that:

$$q = k\cdot A_c \cdot \theta_0 \cdot \left[\frac{h\cdot P}{k\cdot A_c}\right]^{1/2} \cdot e^{-m\cdot x} = \sqrt{h\cdot P\cdot k\cdot A_c} \cdot e^{-m\cdot x} \cdot \theta_0 = M\theta_0 e^{-mx} \tag{26}$$

where $M = \sqrt{hPkA_c}$.

Often we wish to know the total heat flow through the fin, i.e. the heat flow entering at the base (x=0).

$$q = \sqrt{h \cdot P \cdot k \cdot A_c} \cdot \theta_0 = M\theta_0 \tag{27}$$

The insulated tip fin

Assume that the tip is insulated and hence there is no heat transfer:

$$\left. \frac{d\theta}{dx} \right|_{x=L} = 0 \tag{28}$$

The solution to the fin equation is known to be:

$$\theta = C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) \tag{29}$$

Differentiate this expression.

$$\frac{d\theta}{dx} = C \cdot m \cdot \sinh(m \cdot x) + D \cdot m \cdot \cosh(m \cdot x) \tag{30}$$

Apply the first boundary condition at the base:

$$\theta(0) = \theta_0 = C \overset{0}{\sinh(m \cdot 0)} + D \overset{1}{\cosh(m \cdot 0)} \tag{31}$$

So that D = θ_0. Now apply the second boundary condition at the tip to find the value of C:

$$\frac{d\theta}{dx}(L) = 0 = Cm \sinh(m \cdot L) + \theta_0 m \cosh(m \cdot L) \tag{32}$$

which requires that

$$C = -\theta_0 \frac{\cosh(mL)}{\sinh(mL)} \tag{33}$$

This leads to the general temperature profile:

$$\theta(x) = \theta_0 \frac{\cosh m(L - x)}{\cosh(mL)} \tag{34}$$

We may find the heat flow at any value of x by differentiating the temperature profile and substituting it into the Fourier Law:

$$q = -k \cdot A_c \cdot \frac{dT}{dx} = -k \cdot A_c \cdot \frac{d\theta}{dx} \tag{35}$$

So that the energy flowing through the base of the fin is:

$$q = \sqrt{hPkA_c}\,\theta_0 \tanh(mL) = M\theta_0 \tanh(mL) \tag{36}$$

If we compare this result with that for the very long fin, we see that the primary difference in form is in the hyperbolic tangent term. That term, which always results in a number equal to or less than one, represents the reduced heat loss due to the shortening of the fin.

Other tip conditions:
We have already seen two tip conditions, one being the long fin and the other being the insulated tip. Two other possibilities are usually considered for fin analysis: (i) a tip subjected to convective heat transfer, and (ii) a tip with a prescribed temperature. The expressions for temperature distribution and fin heat transfer for all the four cases are summarized in the table below.

Table 3.1

Case	Tip Condition	Temp. Distribution	Fin heat transfer
A	Convection heat transfer: $h\theta(L)=-k(d\theta/dx)_{x=L}$	$\dfrac{\cosh m(L-x)+(h/mk)\sinh m(L-x)}{\cosh mL+(h/mk)\sinh mL}$	$M\theta_o\dfrac{\sinh mL+(h/mk)\cosh mL}{\cosh mL+(h/mk)\sinh mL}$
B	Adiabatic $(d\theta/dx)_{x=L}=0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	$M\theta_0 \tanh mL$
C	Given temperature: $\theta(L)=\theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh m(L-x)+\sinh m(L-x)}{\sinh mL}$	$M\theta_0\dfrac{(\cosh mL-\theta_L/\theta_b)}{\sinh mL}$
D	Infinitely long fin $\theta(L)=0$	e^{-mx}	$M\ \theta_0$

3.4 Fin Effectiveness

How effective a fin can enhance heat transfer is characterized by the fin effectiveness, ε_f, which is as the ratio of fin heat transfer and the heat transfer without the fin. For an adiabatic fin:

$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_C(T_b - T_\infty)} = \frac{\sqrt{hPkA_C}\tanh(mL)}{hA_C} = \sqrt{\frac{kP}{hA_C}}\tanh(mL) \tag{37}$$

If the fin is long enough, mL>2, tanh(mL)→1, and hence it can be considered as infinite fin (case D in Table 3.1). Hence, for long fins,

$$\varepsilon_f \to \sqrt{\frac{kP}{hA_C}} = \sqrt{\left(\frac{k}{h}\right)\frac{P}{A_C}} \tag{38}$$

In order to enhance heat transfer, ε_f should be greater than 1 (In case ε_f <1, the fin would have no purpose as it would serve as an insulator instead). However $\varepsilon_f \geq 2$ is considered unjustifiable because of diminishing returns as fin length increases.

To increase ε_f, the fin's material should have higher thermal conductivity, k. It seems to be counterintuitive that the lower convection coefficient, h, the higher ε_f. Well, if h is very high, it is not necessary to enhance heat transfer by adding heat fins. Therefore, heat fins are more effective if h is low.

Observations:

- If fins are to be used on surfaces separating gas and liquid, fins are usually placed on the gas side. (Why?)
- P/A_C should be as high as possible. Use a square fin with a dimension of W by W as an example: P=4W, $A_C=W^2$, $P/A_C=(4/W)$. The smaller the W, the higher is the P/A_C, and the higher the ε_f.Conclusion: It is preferred to use thin and closely spaced (to increase the total number) fins.

  The effectiveness of a fin can also be characterized by

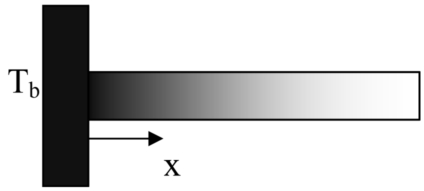
$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_C(T_b - T_\infty)} = \frac{(T_b - T_\infty)/R_{t,f}}{(T_b - T_\infty)/R_{t,h}} = \frac{R_{t,h}}{R_{t,f}} \tag{39}$$

  It is a ratio of the thermal resistance due to convection to the thermal resistance of a fin. In order to enhance heat transfer, the fin's resistance should be lower than the resistance due only to convection.

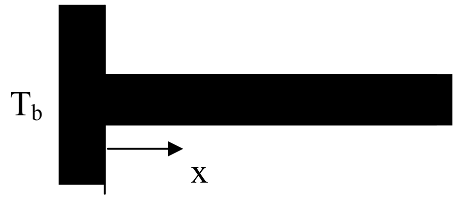
3.5 Fin Efficiency

The fin efficiency is defined as the ratio of the energy transferred through a real fin to that transferred through an ideal fin. An ideal fin is thought to be one made of a perfect or infinite conductor material. A perfect conductor has an infinite thermal conductivity so that the entire fin is at the base material temperature.

$$\eta = \frac{q_{real}}{q_{ideal}} = \frac{\sqrt{h\cdot P\cdot k\cdot A_c}\cdot\theta_L\cdot\tanh(m\cdot L)}{h\cdot(P\cdot L)\cdot\theta_L} \tag{40}$$



Real situation



Ideal situation

Simplifying equation (40):

$$\eta = \sqrt{\frac{k \cdot A_c}{h \cdot P}} \frac{\theta_L \cdot \tanh(m \cdot L)}{L \cdot \theta_L} = \frac{\tanh(m \cdot L)}{m \cdot L} \tag{41}$$

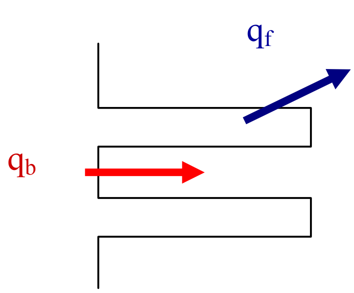
The heat transfer through any fin can now be written as:

$$q.\left[\frac{1}{\eta.h.A_f}\right] = (T - T_\infty \tag{42}$$

The above equation provides us with the concept of fin thermal resistance (using electrical analogy) as

$$R_{t,f} = \frac{1}{\eta.h.A_f} \tag{43}$$

Overall Fin Efficiency:
Overall fin efficiency for an array of fins



Define terms: A_b: base area exposed to coolant

A_f: surface area of a single fin

A_t: total area including base area and total finned surface, $A_t = A_b + NA_f$

N: total number of fins

Heat Transfer from a Fin Array:

$$q_t = q_b + Nq_f = hA_b(T_b - T_\infty) + N\eta_f hA_f(T_b - T_\infty)$$
$$= h[(A_t - NA_f) + N\eta_f A_f](T_b - T_\infty) = h[A_t - NA_f(1-\eta_f)](T_b - T_\infty)$$
$$= hA_t[1 - \frac{NA_f}{A_t}(1-\eta_f)](T_b - T_\infty) = \eta_O hA_t(T_b - T_\infty)$$

Define overall fin efficiency: $\eta_O = 1 - \frac{NA_f}{A_t}(1-\eta_f)$

$$q_t = hA_t\eta_O(T_b - T_\infty) = \frac{T_b - T_\infty}{R_{t,O}} \text{ where } R_{t,O} = \frac{1}{hA_t\eta_O}$$

Compare to heat transfer without fins

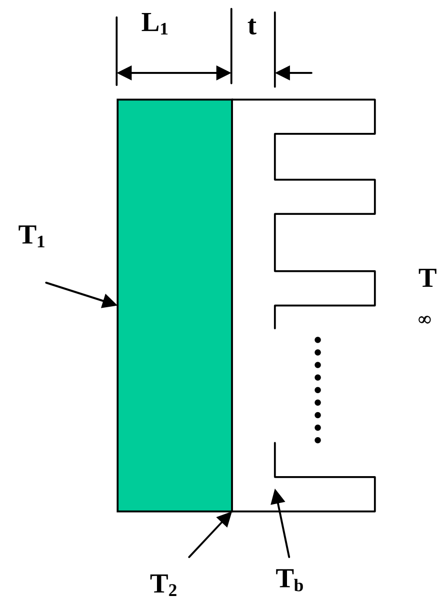
$$q = hA(T_b - T_\infty) = h(A_b + NA_{b,f})(T_b - T_\infty) = \frac{1}{hA}$$

where $A_{b,f}$ is the base area (unexposed) for the fin

To enhance heat transfer $A_t\eta_O >> A$

That is, to increase the effective area $\eta_O A_t$.

Thermal Resistance Concept:



$A = A_b + NA_{b,f}$

$R_b = t/(k_b A)$

$R_1 = L_1/(k_1 A)$

$R_{t,O} = 1/(hA_t\eta_O)$

$$q = \frac{T_1 - T_\infty}{\sum R} = \frac{T_1 - T_\infty}{R_1 + R_b + R_{t,O}}$$