3

Flexural Stresses

Contents

3.1 Introduction

Whenever a horizontal beam is loaded with vertical loads, sometimes, it bends (i.e., deflects) due to the action of the loads. The amount with which a beam bends, depends upon the amount and type of the loads, length of the beam, elasticity of the beam and type of the beam.

The bending moment at a section tends to bend or deflect the beam and the internal stresses resist its bending. The process of bending stops, when every cross-section sets up full resistance to the bending moment. The resistance, offered by the internal stresses, to the bending, is called bending stress, and the relevant theory is called the theory of simple bending.

3.2 Assumptions in the Theory of Simple Bending

The following assumptions are made in the theory of simple bending:

1. The material of the beam is perfectly homogeneous (i.e., of the same kind throughout) and isotropic (i.e., of equal elastic properties in all directions).
2. The beam material is stressed within its elastic limit and thus, obeys Hooke's law.
3. The transverse sections, which were plane before bending, remains plane after bending also.
4. Each layer of the beam is free to expand or contract, independently, of the layer above or below it.
5. The value of E (Young's modulus of elasticity) is the same in tension and compression.
6. The beam is in equilibrium i.e., there is no resultant pull or push in the beam section.

3.3 Theory of Simple Bending

Consider a small length of a simply supported beam subjected to a bending moment as shown in *Fig.3.1* (a). Now consider two sections AB and CD, which are normal to the axis of the beam RS. Due to action of the bending moment, the beam as a whole will bend as shown in *Fig.3.1*(b).

Since we are considering a small length of dx of the beam, therefore the curvature of the beam in this length, is taken to be circular. A little consideration will show that all the layers of the beam, which were originally of the same length do not remain of the same length any more. The top layer of the beam has suffered compression and reduced to A'C'. As we proceed towards the lower layers of the beam, we find that the layers have no doubt suffered compression, but to lesser degree; until we come across the layer RS, which has suffered no change in its length, though bent into R'S'. If we further proceed towards the lower layers, we find the layers have suffered tension, as a result of which the layers are stretched. The amount of extension increases as we proceed lower, until we come across the lowermost layer BD which has been stretched to B' D'.

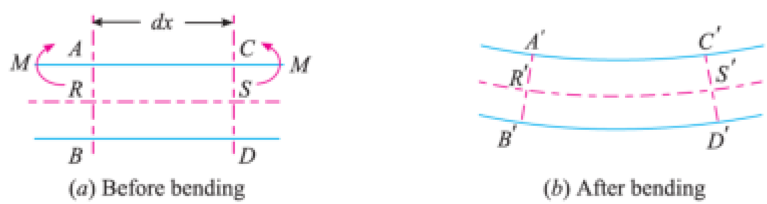

Fig.3.1 – Simple bending

Now we see that the layers above have been compressed and those below RS have been stretched. The amount, by which layer is compressed or stretched, depends upon the position of the layer with reference to RS. This layer RS, which is neither compressed nor stretched, is known as neutral plane or neutral layer. This theory of bending is called ***theory of simple bending***.

3.4 Bending Stress

Consider a small length *dx* of a beam subjected to a bending moment as shown in *Fig.3.2* (a). As a result of this moment, let this small length of beam bend into an arc of a circle with O as centre as shown in *Fig.3.2* (b).

Let M = Moment acting at the beam,

θ = Angle subtended at the centre by the arc

R = Radius of curvature of the beam

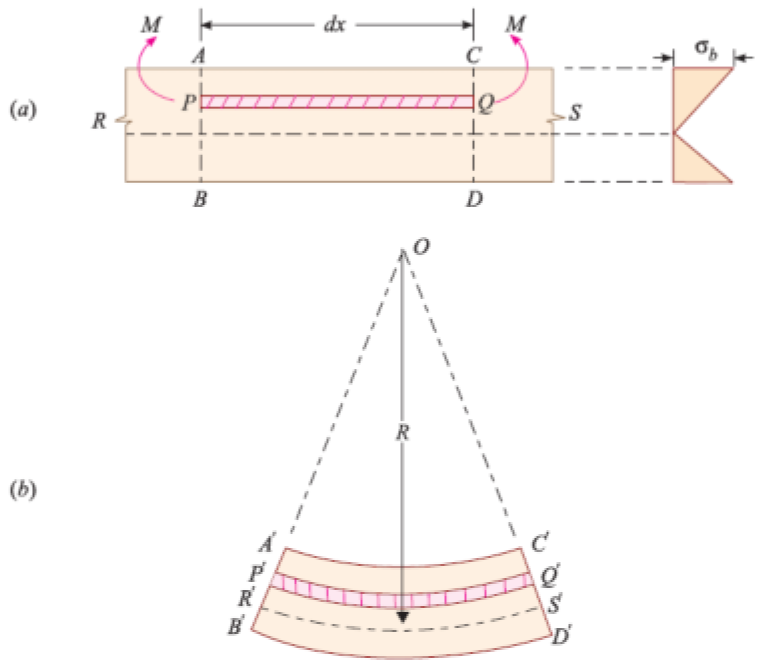

Fig.3.2 – Bending Stress

Now consider a layer PQ at a distance *y* from RS the neutral axis of the beam. Let this layer be compressed to P′ Q′ after bending as shown in *Fig.3.2* (b).

We know that decrease in length of this layer,

$$\delta l = PQ - P'Q'$$

$$Strain\ \varepsilon = \frac{\delta l}{Original\ Length} = \frac{PQ - P'Q'}{PQ}$$

Now from the geometry of the curved beam, we find that the two sections OP′ Q′ and OR′ S′ are similar.

$$\frac{P'Q'}{R'S'} = \frac{R-y}{R}$$

$$1 - \frac{P'Q'}{R'S'} = 1 - \frac{R-y}{R}$$

$$\frac{R'S' - P'Q'}{R'S'} = \frac{y}{R}$$

$$\frac{PQ - P'Q'}{PQ} = \frac{y}{R} \qquad (\because PQ = R'S' = \text{Neutral axis})$$

$$\varepsilon = \frac{y}{R}$$

The strain (ε) of a layer is proportional to its distance from the neutral axis.

The bending stress,

$$\sigma_b = Strain \times Elasticity = \varepsilon \times E$$

$$\sigma_b = \frac{y}{R} \times E$$

Since E and R are constants in this expression, therefore the stress at any point is directly proportional to y, i.e., the distance of the point from the neutral axis. The above expression may also be written as,

$$\boldsymbol{\frac{\sigma_b}{y} = \frac{E}{R} \quad or \quad \sigma_b = \frac{E}{R} \times y}$$

3.5 Position of Neutral Axis

The line of intersection of the neutral layer, with any normal cross-section of a beam, is known as neutral axis of that section. On one side of the neutral axis there are compressive stresses, whereas on the other there are tensile stresses. At the neutral axis, there is no stress of any kind.

Consider a section of the beam as shown in *Fig.3.3*. Let be the neutral axis of the section. Consider a small layer PQ of the beam section at a distance from the neutral axis as shown in *Fig.3.3*.

Let δ a = Area of the layer PQ.

The intensity of stress in the layer PQ,

$$\sigma = y \times \frac{E}{R}$$

∴ Total stress on the layer PQ = Intensity of stress × Area

$$= y \times \frac{E}{R} \times \delta a$$

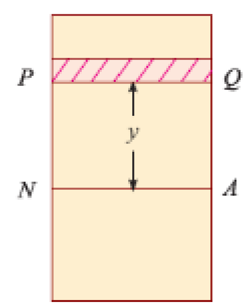

Fig.3.3 – Neutral axis

Total stress of the section.

$$= \Sigma y \times \frac{E}{R} \times \delta a = \frac{E}{R}.\Sigma y.\delta a$$

Since the section is in equilibrium, therefore total stress, from top to bottom, must be equal to zero.

$$\frac{E}{R}.\Sigma y.\delta a = 0$$

$$\Sigma y.\delta a = 0 \qquad (\because \text{E/R cannot be equal to zero})$$

A little consideration will show that y × δa is the moment of the area about the neutral axis and Σy × δa is the moment of the entire area of the cross-section about the neutral axis. It is thus obvious that the neutral axis of the section will be so located that moment of the entire area about the axis is zero. We know that the moment of any area about an axis passing through its central axis of a section always passes through its centroid. Thus to locate the neutral axis of a section, first find out the centroid of the section and then draw a line passing through this centroid and normal to the plane of bending. This line will be the neutral axis of the section.

3.6 Moment of Resistance

On one side of the neutral axis there are compressive stresses and on the other there are tensile stresses. These stresses form a couple, whose moment must be equal to the external moment (M). The moment of this couple, which resists the external bending moment, is known as ***moment of resistance.***

Consider a section of the beam as shown in *Fig.3.4*. Let NA be the neutral axis of the section. Now consider a small layer PQ of the beam section at a distance y from the neutral axis as shown in *Fig.3.4*.

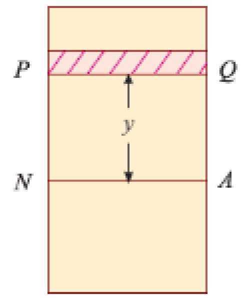

Fig.3.4 –Moment of Resistance

Let δa = Area of the layer PQ.

The intensity of stress in the layer PQ,

$$\sigma = y \times \frac{E}{R}$$

∴ Total stress in the layer PQ

$$= y \times \frac{E}{R} \times \delta a$$

Moment of this total stress about the neutral axis

$$= y \times \frac{E}{R} \times \delta a \times y = \frac{E}{R} . y^2 . \delta a$$

The algebraic sum of all such moments about the neutral axis must be equal to M. Therefore

$$M = \Sigma \frac{E}{R} . y^2 . \delta a = \frac{E}{R} \Sigma y^2 . \delta a$$

The expression $\Sigma y^2 . \delta a$ represents the moment of inertia of the area of the whole section about the neutral axis. Therefore

$$M = \frac{E}{R} \times I$$

where *I* = moment of inertia

$$\frac{M}{I} = \frac{E}{R}$$

We have already discussed

$$\frac{\sigma}{y} = \frac{E}{R}$$

$$\boldsymbol{\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}}$$

It is the most important equation in the theory of simple bending, which gives us relation between various characteristics of a beam.

3.7 Section Modulus

The relation for finding out the bending stress on the extreme fibre of a section, i.e.,

$$\frac{M}{I} = \frac{\sigma}{y}$$

From this relation, we find that the stress in a fibre is proportional to its distance from the c.g. If y_{max} is the distance between the c.g. of the section and the extreme fibre of the stress, then

$$M = \sigma_{max} \times \frac{I}{y_{max}} = \sigma_{max} \times Z$$

where Z = I/y_{max}. The term 'Z' is known as ***modulus of section or section modulus***. The general practice of writing the above equation is M = σ × Z, where σ denotes the maximum stress, tensile or compressive in nature.

We know that if the section of a beam to, is symmetrical, its centre of gravity and hence the neutral axis will lie at the middle of its depth. We shall now consider the modulus of section of the following sections:

1. Rectangular section. 2. Circular section.

1. **Rectangular section**

We know that moment of inertia of a rectangular section about an axis through its centre of gravity.

$$I = \frac{bd^3}{12}$$

$$\therefore \textit{Moduluds of section,} \qquad Z = \frac{I}{y} = \frac{bd^3}{12} \times \frac{2}{d}$$

$$Z = \frac{bd^2}{6}$$

2. **Circular section**

We know that moment of inertia of a circular section about an axis through its c.g.,

$$I = \frac{\pi}{64} d^4$$

$$\therefore \textit{Moduluds of section,} \qquad Z = \frac{I}{y} = \frac{\pi}{64} d^4 \times \frac{2}{d}$$

$$Z = \frac{\pi}{32} d^3$$

3.8 Strength of a Section

It is also termed as flexural strength of a section, which means the moment of resistance offered by it. We have already discussed the relations :

$$\frac{M}{I} = \frac{\sigma}{y} \qquad or \qquad M = \frac{\sigma}{y} \times I \qquad and \qquad M = \sigma Z$$

It is thus obvious that the moment of resistance depends upon moment of inertia (or section modulus) of the section. A little consideration will show that the moment of inertia of beam section does not depend upon its cross-section area, but its disposition in relation to the neutral axis.

We know that in the case of a beam, subjected to transverse loading, the bending stress at a point is directly proportional to its distance from the neutral axis. It is thus obvious that a larger area near the neutral axis of a beam is uneconomical. This idea is put into practice, by providing beams of section, where the flanges alone withstand almost all the bending stress.