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Results on Rational Type Contraction in Partial Order Metric Spaces

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Abstract: In this study, we have proved certain common fixed point theorems for rational type contractions in partial order metric space, which are connected to integral type equations. In addition, we provided an example for verifying our key findings.

1 INTRODUCTION

Fixed point theory is one of the most important topics of mathematics, with numerous applications in several aspects of physical science as well as various domains of mathematics. The Banach contraction principle has a huge number of generalizations like Ciric contraction, S-contraction, Chatterjee contraction, weak contractions principle etc. These contractions proved many results in analysis and play a central role to find the existence and uniqueness of the results in analysis and many other different fields of mathematics. In partially ordered sets, Ran and Reuing [1] generalized the Banach contraction principle and applied it to linear and nonlinear matrix equations in 2003. Nieto and Lopez [6, 7, 8] expanded these findings in 2005 and used them to find the unique solution to first-order differential equations. After that several authors [2, 9, 10, 11, 13, 15, 17] investigated and extended many results.

Here, we have generalized some results in rational type contraction of Chandok [14] and some other papers [4, 5, 12, 16]. We've also provided an example to demonstrate the existence of our major conclusions and their applications to integral type problems.

2 PRELIMINARIES AND DEFINITIONS

2.1 Coincidence and common fixed point: [14] Consider the metric space $(\mathfrak{R}, \delta)$ and the non empty subset $\mathbb{M}$ of $\mathfrak{R}$. If

$$\xi\alpha = \zeta\alpha \quad (\xi\alpha = \zeta\alpha = \alpha),$$

then a point $\alpha \in \mathbb{M}$ is known as coincidence point (common fixed point) of two self mappings ξ and ζ respectively. $C(\xi, \zeta)$ and $F(\xi, \zeta)$ are the sets of coincidence points and common fixed points, respectively.

2.2 Commuting and compatible mappings: [14] Two self mappings $\xi, \zeta: \mathbb{M} \to \mathbb{M}$ are known as commuting, if for all $\alpha \in \mathbb{M}$ such that

$$\zeta\xi\alpha = \xi\zeta\alpha$$

and known as compatible, if for all $\alpha \in \mathbb{M}$ such that

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$$\lim_{n\to\infty} \delta(\zeta\xi\alpha_n, \xi\zeta\alpha_n) = 0,$$

whenever the sequence $\{\alpha_n\}$ satisfying

$$\lim_{n\to\infty} \xi\alpha_n = \lim_{n\to\infty} \zeta\alpha_n = p$$

for some $p \in \mathbb{M}$. If they commute at their coincidence points, they are said to be weakly compatible, i.e.,

$$\zeta\xi\alpha = \xi\zeta\alpha.$$

2.3 Strictly increasing and decreasing mappings: A self mapping $\zeta: \mathfrak{R} \to \mathfrak{R}$ in a partially ordered set $(\mathfrak{R}, \leq)$ is said to be strictly increasing if $\zeta(\alpha) < \zeta(\beta)$ for all $\alpha, \beta \in \mathfrak{R}$ with $\alpha < \beta$, and strictly decreasing if $\zeta(\alpha) > \zeta(\beta)$ for all $\alpha, \beta \in \mathfrak{R}$ with $\alpha < \beta$.

2.4 Mixed monotone mappings:[14] Two self mappings $\xi, \zeta: \mathbb{M} \to \mathbb{M}$ on a partially ordered set $(\mathbb{M}, \leq)$ is said to be ζ monotone ξ-nondecreasing, if satisfying the condition

$$\xi\alpha \leq \xi\nu \Rightarrow \zeta\alpha \leq \zeta\nu$$

for all $\alpha, \nu \in \mathbb{M}$. If ξ is identity mapping, then ζ is monotone nondecreasing mapping.

Similarly, ζ is said to be monotone ξ-nonincreasing, if satisfying the condition

$$\xi\alpha \leq \xi\nu \Rightarrow \zeta\alpha \geq \zeta\nu$$

for all $\alpha, \nu \in \mathbb{M}$.

3 MAIN RESULTS

Theorem 3.1 Let $(\mathfrak{R}, \delta)$ be a complete metric space endowed with partial order set in $\mathfrak{R}$ and two continuous compatible self mappings ζ and ξ on $\mathfrak{R}$ with $\zeta(\mathfrak{R}) \subseteq \xi(\mathfrak{R})$. If ζ is a monotone ξ-nondecreasing and satisfy the contraction

$$\begin{aligned}\delta(\zeta\alpha, \zeta\nu) \leq \mathbb{K}\left[\frac{\delta(\xi\alpha,\zeta\alpha)\delta(\xi\nu,\zeta\nu)}{\delta(\xi\alpha,\xi\nu)}\right] + \mathbb{L}\left[\frac{\delta(\xi\alpha,\zeta\alpha)[\delta(\xi\nu,\zeta\nu)+1]}{\delta(\xi\alpha,\xi\nu)+1}\right] \\ +\mathbb{Q}[\delta(\xi\alpha, \xi\nu)]\end{aligned} \tag{3.1}$$

for all $\alpha, \nu \in \mathfrak{R}$ and $\mathbb{K}, \mathbb{L}, \mathbb{Q} \in [0,1)$ with $\mathbb{K} + \mathbb{L} + \mathbb{Q} < 1$. Then ζ and ξ have a coincidence point, if there exist $\alpha_0 \in \mathfrak{R}$ such that $\xi(\alpha_0) \leq \zeta(\alpha_0)$. Furthermore, ξ and ζ have a unique common fixed point if they are weakly compatible mappings in $\mathfrak{R}$.

Proof: Let $\alpha_0 \in \mathfrak{R}$ be such that $\xi(\alpha_0) \leq \zeta(\alpha_0)$. Since $\zeta(\mathfrak{R}) \subseteq \xi(\mathfrak{R})$, we can choose $\alpha_1 \in \mathfrak{R}$ such that $\xi\alpha_1 = \zeta\alpha_0$. Since $\zeta\alpha_1 \in \xi(\mathfrak{R})$, there exist $\alpha_2 \in \mathfrak{R}$, such that $\xi\alpha_2 = \zeta\alpha_1$.
By induction, we can construct a sequence $\{\alpha_n\}$ in $\mathfrak{R}$ such that $\xi\alpha_{n+1} = \zeta\alpha_n$ for every $n \geq 0$.
By above hypothesis, we have $\xi(\alpha_0) \leq \zeta(\alpha_0) = \xi(\alpha_1)$. Since ζ is a monotone ξ-nondecreasing mappings, then

$$\zeta(\alpha_0) \leq \zeta(\alpha_1). \tag{3.2}$$

Similarly, we continuing, we have

$$\zeta(\alpha_0) \leq \zeta(\alpha_1) \leq \zeta(\alpha_2) \leq \ldots\ldots\ldots \leq \zeta(\alpha_n) \leq \zeta(\alpha_{n+1}) \leq \ldots\ldots \tag{3.3}$$

and

$$\xi(\alpha_0) \leq \xi(\alpha_1) \leq \xi(\alpha_2) \leq \ldots\ldots\ldots \leq \xi(\alpha_n) \leq \xi(\alpha_{n+1}) \leq \ldots\ldots \tag{3.4}$$

Suppose that $\delta(\zeta\alpha_{n+1}, \alpha_n) = 0$ for all n, then $\zeta(\alpha_{n+1}) = \zeta(\alpha_n)$.

Since $\zeta(\alpha_n) = \xi(\alpha_{n+1})$, therefore $\zeta(\alpha_{n+1}) = \xi(\alpha_{n+1})$ for all n.
Then α_{n+1} for all n, are coincidence point of ζ and ξ.
Again, we suppose that $\delta(\zeta\alpha_{n+1}, \alpha_n) > 0 \forall$ n, then by equation (3.1)

$$\delta(\zeta\alpha_{n+1}, \zeta\alpha_n) \le \mathbb{K}[\frac{\delta(\xi\alpha_{n+1},\zeta\alpha_{n+1})\delta(\xi\alpha_n,\zeta\alpha_n)}{\delta(\xi\alpha_{n+1},\xi\alpha_n)}] + \mathbb{L}[\frac{\delta(\xi\alpha_{n+1},\zeta\alpha_{n+1})[\delta(\xi\alpha_n,\zeta\alpha_n)+1]}{\delta(\xi\alpha_{n+1},\xi\alpha_n)+1}] + \mathbb{Q}[\delta(\xi\alpha_{n+1}, \xi\alpha_n)]$$

$$\delta(\zeta\alpha_{n+1}, \zeta\alpha_n) \le \mathbb{K}[\frac{\delta(\zeta\alpha_n,\zeta\alpha_{n+1})\delta(\zeta\alpha_{n-1},\zeta\alpha_n)}{\delta(\zeta\alpha_n,\zeta\alpha_{n-1})}] + \mathbb{L}[\frac{\delta(\zeta\alpha_n,\zeta\alpha_{n+1})[\delta(\zeta\alpha_{n-1},\zeta\alpha_n)+1]}{\delta(\zeta\alpha_n,\zeta\alpha_{n-1})+1}] + \mathbb{Q}[\delta(\zeta\alpha_n, \zeta\alpha_{n-1})]$$

$$\le \mathbb{K}\delta(\zeta\alpha_{n+1}, \zeta\alpha_n) + \mathbb{L}\delta(\zeta\alpha_{n+1}, \zeta\alpha_n) + \mathbb{Q}\delta(\zeta\alpha_n, \zeta\alpha_{n-1})$$

$$(1 - \mathbb{K} - \mathbb{L})\delta(\zeta\alpha_{n+1}, \zeta\alpha_n) \le \mathbb{Q}\delta(\zeta\alpha_n, \zeta\alpha_{n-1})$$

$$\delta(\zeta\alpha_{n+1}, \zeta\alpha_n) \le (\frac{\mathbb{Q}}{1-\mathbb{K}-\mathbb{L}})\delta(\zeta\alpha_n, \zeta\alpha_{n-1}). \tag{3.5}$$

Continuing this process, we obtain

$$\delta(\zeta\alpha_{n+1}, \zeta\alpha_n) \le (\frac{\mathbb{Q}}{1-\mathbb{K}-\mathbb{L}})^n\delta(\zeta\alpha_1, \zeta\alpha_0). \tag{3.6}$$

Taking $\lambda = \frac{\mathbb{Q}}{1-\mathbb{K}-\mathbb{L}} < 1$, we have

$$\delta(\zeta\alpha_{n+1}, \zeta\alpha_n) \le (\lambda)^n\delta(\zeta\alpha_1, \zeta\alpha_0). \tag{3.7}$$

We'll now prove that $\{\zeta\alpha_n\}$ is a Cauchy sequence. For $m \ge n$, we have

$$\delta(\zeta\alpha_m, \zeta\alpha_n) \le \delta(\zeta\alpha_m, \zeta\alpha_{m-1}) + \delta(\zeta\alpha_{m-1}, \zeta\alpha_{m-2}) + \ldots\ldots + \delta(\zeta\alpha_{n+1}, \zeta\alpha_n)$$

$$\le (\lambda^{m-1} + \lambda^{m-2} + \ldots. + \lambda^n)\delta(\zeta\alpha_1, \zeta\alpha_0)$$

$$= (\frac{\lambda^n(1-\lambda^{m-n-1})}{1-\lambda})\delta(\zeta\alpha_1, \zeta\alpha_0), \tag{3.8}$$

taking $m, n \to \infty$, we have $\delta(\zeta\alpha_m, \zeta\alpha_n) \to 0$.
Thus, in a complete metric space $\mathfrak{R}$, the sequence $\{\zeta\alpha_n\}$ is a Cauchy sequence. Therefore there exist $\omega \in \mathfrak{R}$, such that

$$\lim_{n\to\infty} \zeta\alpha_n = \omega. \tag{3.9}$$

By continuity of ζ, we have

$$\lim_{n\to\infty} \zeta(\zeta\alpha_n) = \zeta\omega. \tag{3.10}$$

Since $\xi\alpha_{n+1} = \zeta\alpha_n$, therefore

$$\lim_{n\to\infty} \xi\alpha_{n+1} = \lim_{n\to\infty} \zeta\alpha_n = \zeta\omega. \tag{3.11}$$

Since ζ and ξ are compatible, then

$$\lim_{n\to\infty} \delta(\xi(\zeta\alpha_n), \zeta(\xi\alpha_n)) = 0. \tag{3.12}$$

By triangular inequality of δ, we have

$$\delta(\zeta\omega, \xi\omega) \le \delta(\zeta\omega, \zeta(\xi\alpha_n)) +. \delta(\zeta(\xi\alpha_n), \xi(\zeta\alpha_n)) + \delta(\xi(\zeta\alpha_n), \xi\omega). \tag{3.13}$$

Taking $n \to \infty$ and using (3.10), (3.11) and (3.12) in (3.13), we have

$$\delta(\zeta\omega, \xi\omega) = 0 \tag{3.14}$$

$$\zeta\omega = \xi\omega. \tag{3.15}$$

Hence ω is a coincidence point of ζ and ξ, i.e. existence of coincidence point for ζ and ξ.
Now, we assume that both mappings ζ and ξ are weakly compatible. Consider ω be the coincidence point of ζ and ξ and choose

$$\omega = \xi z = \zeta z \tag{3.16}$$

for some $z \in \mathfrak{R}$, then

$$\zeta\omega = \zeta\xi z = \zeta(\zeta z) \tag{3.17}$$

and

$$\xi\omega = \xi(\xi z) = \xi\zeta z. \tag{3.18}$$

Therefore, from (3.16), (3.17), and (3.18) we obtain

$$\zeta\omega = \xi\omega = \zeta\xi z = \xi\zeta z. \tag{3.19}$$

From (3.1), we obtain

$$\begin{aligned}\delta(\zeta z, \zeta\omega) &\leq \mathbb{K}[\tfrac{\delta(\xi z,\zeta z)\delta(\xi\omega,\zeta\omega)}{\delta(\xi z,\xi\omega)}] + \mathbb{L}[\tfrac{\delta(\xi z,\zeta z)[\delta(\xi\omega,\zeta\omega)+1]}{\delta(\xi z,\xi\omega)+1}] \\ &\quad + \mathbb{Q}[\delta(\xi z, \xi\omega)] \\ &= \mathbb{K}[\tfrac{\delta(\omega,\omega)\delta(\xi\omega,\zeta\omega)}{\delta(\xi z,\xi\omega)}] + \mathbb{L}[\tfrac{\delta(\omega,\omega)[\delta(\xi\omega,\zeta\omega)+1]}{\delta(\xi z,\xi\omega)+1}] + \mathbb{Q}[\delta(\xi z, \xi\omega)] \\ &= \mathbb{Q}[\delta(\xi z, \xi\omega)]\end{aligned}$$

$$\delta(\zeta z, \zeta\omega) \leq \mathbb{Q}[\delta(\xi z, \xi\omega)]. \tag{3.20}$$

Since $\mathbb{Q} < 1$ and using equations (3.16) and (3.19), we have

$$\delta(\zeta z, \zeta\omega) < \delta(\xi z, \xi\omega) = \delta(\zeta z, \zeta\omega). \tag{3.21}$$

$$\delta(\zeta z, \zeta\omega) = 0 \Rightarrow \zeta z = \zeta\omega. \tag{3.22}$$

Therefore,

$$\zeta\omega = \xi\omega = \omega. \tag{3.23}$$

Hence ω is a common fixed point of ζ and ξ.
Now, we have to prove that ζ and ξ have a unique common fixed point. Assume that ω and $\mu(\omega \neq \mu)$ are two separate common fixed points of ζ and ξ, such that

$$\zeta\omega = \xi\omega = \omega \quad \text{and} \quad \zeta\mu = \xi\mu = \mu.$$

From (3.1), we have

$$\begin{aligned}\delta(\omega, \mu) = \delta(\zeta\omega, \zeta\mu) &\leq \mathbb{K}[\tfrac{\delta(\xi\omega,\zeta\omega)\delta(\xi\mu,\zeta\mu)}{\delta(\xi\omega,\xi\mu)}] \\ &\quad + \mathbb{L}[\tfrac{\delta(\xi\omega,\zeta\omega)[\delta(\xi\mu,\zeta\mu)+1]}{\delta(\xi\omega,\xi\mu)+1}] + \mathbb{Q}[\delta(\xi\omega, \xi\mu)] \\ &= \mathbb{K}[\tfrac{\delta(\omega,\omega)\delta(\mu,\mu)}{\delta(\omega,\mu)}] + \mathbb{L}[\tfrac{\delta(\omega,\omega)[\delta(\mu,\mu)+1]}{\delta(\omega,\mu)+1}] + \mathbb{Q}[\delta(\omega, \mu)] \\ &= \mathbb{Q}\delta(\omega, \mu),\end{aligned}$$

$$\delta(\omega,\mu) = \mathbb{Q}\delta(\omega,\mu). \tag{3.24}$$

Since $\mathbb{Q} < 1$, therefore

$$\delta(\omega,\mu) < \delta(\omega,\mu). \tag{3.25}$$

As a basis of this contradiction, our assumption is incorrect. Therefore the fixed point is unique.

□

Corollary 3.2 Let $(\mathfrak{R},\delta)$ be a complete metric space endowed with partial order set in $\mathfrak{R}$ and two continuous compatible self mappings ζ and ξ on $\mathfrak{R}$ with $\zeta(\mathfrak{R}) \subseteq \xi(\mathfrak{R})$. If ζ is a monotone ξ-nondecreasing and satisfy the contraction

$$\begin{aligned}\delta(\zeta\alpha,\zeta\nu) \leq \mathbb{K}[\tfrac{\delta(\xi\alpha,\zeta\alpha)\delta(\xi\nu,\zeta\nu)}{\delta(\xi\alpha,\xi\nu)}] + \mathbb{L}[\tfrac{\delta(\xi\alpha,\zeta\alpha)[\delta(\xi\nu,\zeta\nu)+1]}{\delta(\xi\alpha,\xi\nu)+1}] \\ +\mathbb{P}[\delta(\xi\alpha,\zeta\nu) + \delta(\xi\nu,\zeta\alpha)] + \mathbb{Q}[\delta(\xi\alpha,\xi\nu)]\end{aligned} \tag{3.26}$$

for all $\alpha,\nu \in \mathfrak{R}$ and $\mathbb{K},\mathbb{L},\mathbb{P},\mathbb{Q} \in [0,1)$ with $\mathbb{K}+\mathbb{L}+\mathbb{P}+\mathbb{Q} < 1$. Then ζ and ξ have a coincidence point, if there exist $\alpha_0 \in \mathfrak{R}$ such that $\xi(\alpha_0) \leq \zeta(\alpha_0)$. Furthermore, ξ and ζ have a unique common fixed point if they are weakly compatible mappings in $\mathfrak{R}$.

□

Corollary 3.3 Let $(\mathfrak{R},\delta)$ be a complete metric space endowed with partial order set in $\mathfrak{R}$ and two continuous compatible self mappings ζ and ξ on $\mathfrak{R}$ with $\zeta(\mathfrak{R}) \subseteq \xi(\mathfrak{R})$. If ζ is a monotone ξ-nondecreasing and satisfy the contraction

$$\begin{aligned}\delta(\zeta\alpha,\zeta\nu) \leq \mathbb{K}[\tfrac{\delta(\xi\alpha,\zeta\alpha)\delta(\xi\nu,\zeta\nu)}{\delta(\xi\alpha,\xi\nu)}] \\ +\mathbb{L}[\delta(\xi\alpha,\zeta\nu) + \delta(\xi\nu,\zeta\alpha)] + \mathbb{Q}[\delta(\xi\alpha,\xi\nu)]\end{aligned} \tag{3.27}$$

for all $\alpha,\nu \in \mathfrak{R}$ and $\mathbb{K},\mathbb{L},\mathbb{Q} \in [0,1)$ with $\mathbb{K}+\mathbb{L}+\mathbb{Q} < 1$. Then ζ and ξ have a coincidence point, if there exist $\alpha_0 \in \mathfrak{R}$ such that $\xi(\alpha_0) \leq \zeta(\alpha_0)$. Furthermore, ξ and ζ have a unique common fixed point if they are weakly compatible mappings in $\mathfrak{R}$.

□

Corollary 3.4 Let $(\mathfrak{R},\delta)$ be a complete metric space endowed with partial order set in $\mathfrak{R}$ and two continuous compatible self mappings ζ and ξ on $\mathfrak{R}$ with $\zeta(\mathfrak{R}) \subseteq \xi(\mathfrak{R})$. If ζ is a monotone ξ-nondecreasing and satisfy the contraction

$$\begin{aligned}\delta(\zeta\alpha,\zeta\nu) \leq \mathbb{K}[\tfrac{\delta(\xi\alpha,\zeta\alpha)[\delta(\xi\nu,\zeta\nu)+1]}{\delta(\xi\alpha,\xi\nu)+1}] \\ +\mathbb{L}[\delta(\xi\alpha,\zeta\alpha) + \delta(\xi\nu,\zeta\nu)] + \mathbb{Q}[\delta(\xi\alpha,\xi\nu)]\end{aligned} \tag{3.28}$$

for all $\alpha,\nu \in \mathfrak{R}$ and $\mathbb{K},\mathbb{L},\mathbb{Q} \in [0,1)$ with $\mathbb{K}+\mathbb{L}+\mathbb{Q} < 1$. Then ζ and ξ have a coincidence point, if there exist $\alpha_0 \in \mathfrak{R}$ such that $\xi(\alpha_0) \leq \zeta(\alpha_0)$. Furthermore, ξ and ζ have a unique common fixed point if they are weakly compatible mappings in $\mathfrak{R}$.

□

Corollary 3.5[14] Let $(\mathfrak{R},\delta)$ be a complete metric space endowed with partial order set in $\mathfrak{R}$ and two continuous compatible self mappings ζ and ξ on $\mathfrak{R}$ with $\zeta(\mathfrak{R}) \subseteq \xi(\mathfrak{R})$. If ζ is a monotone ξ-nondecreasing and satisfy the contraction

$$\delta(\zeta\alpha,\zeta\nu) \leq \mathbb{K}[\tfrac{\delta(\xi\alpha,\zeta\alpha)\delta(\xi\nu,\zeta\nu)}{\delta(\xi\alpha,\xi\nu)}] + \mathbb{L}[\delta(\xi\alpha,\xi\nu)] \tag{3.29}$$

for all $\alpha,\nu \in \mathfrak{R}$ and $\mathbb{K},\mathbb{L} \in [0,1)$ with $\mathbb{K}+\mathbb{L} < 1$. Then ζ and ξ have a coincidence point, if there exist $\alpha_0 \in \mathfrak{R}$ such that $\xi(\alpha_0) \leq \zeta(\alpha_0)$. Furthermore, ξ and ζ have a unique common fixed point if they are weakly compatible mappings in $\mathfrak{R}$.

□

Remark: If $\xi = I$ (Identity mapping) in above theorem, we have results of Harjani, Lopez and Sadarangani [3], such that ζ has a unique fixed point in each above results .
Above contraction mapping defined in equation (3.1) and all corollaries can be relate to integral equations.
Let Λ be the set of functions $\chi: [0, \infty) \to [0, \infty)$ satisfying the following hypothesis

(i) χ is a Lebesgue integrable mapping on each compact subset of $[0, \infty)$;

(ii) for any $\epsilon > 0$, we have $\int_0^\epsilon \chi(s)ds > 0$

Corollary 3.6 Let $(\Re, \delta)$ be a complete metric space endowed with partial order set in $\Re$ and two continuous compatible self mappings ζ and ξ on $\Re$ with $\zeta(\Re) \subseteq \xi(\Re)$. If ζ is a monotone ξ-nondecreasing and satisfy the integral type contraction

$$\int_0^{\delta(\zeta\alpha,\zeta\nu)} \chi(s)ds \leq \mathbb{K}\int_0^{\frac{\delta(\xi\alpha,\zeta\alpha)\delta(\xi\nu,\zeta\nu)}{\delta(\xi\alpha,\xi\nu)}} \chi(s)ds + \mathbb{L}\int_0^{\frac{\delta(\xi\alpha,\zeta\alpha)[\delta(\xi\nu,\zeta\nu)+1]}{\delta(\xi\alpha,\xi\nu)+1}} \chi(s)ds + \mathbb{Q}\int_0^{\delta(\xi\alpha,\xi\nu)} \chi(s)ds \tag{3.30}$$

for all $\alpha, \nu \in \Re$ and $\mathbb{K}, \mathbb{L}, \mathbb{Q} \in [0,1)$ with $\mathbb{K} + \mathbb{L} + \mathbb{Q} < 1$. Then ζ and ξ have a coincidence point, if there exist $\alpha_0 \in \Re$ such that $\xi(\alpha_0) \leq \zeta(\alpha_0)$. Furthermore, ξ and ζ have a unique common fixed point if they are weakly compatible mappings in $\Re$.

□

Corollary 3.7 Let $(\Re, \delta)$ be a complete metric space endowed with partial order set in $\Re$ and two continuous compatible self mappings ζ and ξ on $\Re$ with $\zeta(\Re) \subseteq \xi(\Re)$. If ζ is a monotone ξ-nondecreasing and satisfy the integral type contraction

$$\int_0^{\delta(\zeta\alpha,\zeta\nu)} \chi(s)ds \leq \mathbb{K}\int_0^{\frac{\delta(\xi\alpha,\zeta\alpha)\delta(\xi\nu,\zeta\nu)}{\delta(\xi\alpha,\xi\nu)}} \chi(s)ds + \mathbb{L}\int_0^{\frac{\delta(\xi\alpha,\zeta\alpha)[\delta(\xi\nu,\zeta\nu)+1]}{\delta(\xi\alpha,\xi\nu)+1}} \chi(s)ds + \mathbb{P}\int_0^{\delta(\xi\alpha,\zeta\nu)+\delta(\xi\nu,\zeta\alpha)} \chi(s) + \mathbb{Q}\int_0^{\delta(\xi\alpha,\xi\nu)} \chi(s)ds \tag{3.31}$$

for all $\alpha, \nu \in \Re$ and $\mathbb{K}, \mathbb{L}, \mathbb{P}, \mathbb{Q} \in [0,1)$ with $\mathbb{K} + \mathbb{L} + \mathbb{P} + \mathbb{Q} < 1$. Then ζ and ξ have a coincidence point, if there exist $\alpha_0 \in \Re$ such that $\xi(\alpha_0) \leq \zeta(\alpha_0)$. Furthermore, ξ and ζ have a unique common fixed point if they are weakly compatible mappings in $\Re$.

□

Corollary 3.8 Let $(\Re, \delta)$ be a complete metric space endowed with partial order set in $\Re$ and two continuous compatible self mappings ζ and ξ on $\Re$ with $\zeta(\Re) \subseteq \xi(\Re)$. If ζ is a monotone ξ-nondecreasing and satisfy the integral type contraction

$$\int_0^{\delta(\zeta\alpha,\zeta\nu)} \chi(s)ds \leq \mathbb{K}\int_0^{\frac{\delta(\xi\alpha,\zeta\alpha)\delta(\xi\nu,\zeta\nu)}{\delta(\xi\alpha,\xi\nu)}} \chi(s)ds + \mathbb{L}\int_0^{\delta(\xi\alpha,\zeta\nu)+\delta(\xi\nu,\zeta\alpha)} \chi(s)ds + \mathbb{Q}\int_0^{\delta(\xi\alpha,\xi\nu)} \chi(s)ds \tag{3.32}$$

for all $\alpha, \nu \in \Re$ and $\mathbb{K}, \mathbb{L}, \mathbb{Q} \in [0,1)$ with $\mathbb{K} + \mathbb{L} + \mathbb{Q} < 1$. Then ζ and ξ have a coincidence point, if there exist $\alpha_0 \in \Re$ such that $\xi(\alpha_0) \leq \zeta(\alpha_0)$. Furthermore, ξ and ζ have a unique common fixed point if they are weakly compatible mappings in $\Re$.

□

Corollary 3.9 Let $(\Re, \delta)$ be a complete metric space endowed with partial order set in $\Re$ and two continuous compatible self mappings ζ and ξ on $\Re$ with $\zeta(\Re) \subseteq \xi(\Re)$. If ζ is a monotone ξ-nondecreasing and satisfy the integral type contraction

$$\int_0^{\delta(\zeta\alpha,\zeta\nu)} \chi(s)ds \leq \mathbb{K}\int_0^{\frac{\delta(\xi\alpha,\zeta\alpha)[\delta(\xi\nu,\zeta\nu)+1]}{\delta(\xi\alpha,\xi\nu)+1}} \chi(s)ds + \mathbb{L}\int_0^{\delta(\xi\alpha,\zeta\alpha)+\delta(\xi\nu,\zeta\nu)} \chi(s)ds + \mathbb{Q}\int_0^{\delta(\xi\alpha,\xi\nu)} \chi(s)ds \tag{3.33}$$

for all $\alpha, \nu \in \Re$ and $\mathbb{K}, \mathbb{L}, \mathbb{Q} \in [0,1)$ with $\mathbb{K} + \mathbb{L} + \mathbb{Q} < 1$. Then ζ and ξ have a coincidence point, if there exist $\alpha_0 \in \Re$ such that $\xi(\alpha_0) \leq \zeta(\alpha_0)$. Furthermore, ξ and ζ have a unique common fixed point if they are weakly compatible mappings in $\Re$.

□

Corollary 3.10 Let $(\Re,\delta)$ be a complete metric space endowed with partial order set in $\Re$ and two continuous compatible self mappings ζ and ξ on $\Re$ with $\zeta(\Re)\subseteq\xi(\Re)$. If ζ is a monotone ξ-nondecreasing and satisfy the integral type contraction

$$\int_0^{\delta(\zeta\alpha,\zeta\nu)}\chi(s)ds\le\mathbb{K}\int_0^{\frac{\delta(\xi\alpha,\zeta\alpha)\delta(\xi\nu,\zeta\nu)}{\delta(\xi\alpha,\xi\nu)}}\chi(s)ds+\mathbb{L}\int_0^{\delta(\xi\alpha,\xi\nu)}\chi(s)ds \tag{3.34}$$

for all $\alpha,\nu\in\Re$ and $\mathbb{K},\mathbb{L}\in[0,1)$ with $\mathbb{K}+\mathbb{L}<1$. Then ζ and ξ have a coincidence point, if there exist $\alpha_0\in\Re$ such that $\xi(\alpha_0)\le\zeta(\alpha_0)$. Furthermore, ξ and ζ have a unique common fixed point if they are weakly compatible mappings in $\Re$.

□

Example 3.1 Consider metric $\delta:\Re\times\Re\to[0,\infty)$ defined by $\delta(\alpha,\nu)=|\alpha-\nu|$, where $\Re=[0,1]$ with usual partial order $\le$. Let ζ and ξ be two self mappings on $\Re$, such that $\zeta(\alpha)=\frac{\alpha^2}{3}$ and $\xi(\alpha)=\frac{3\alpha^2}{1+\alpha}$, then ζ and ξ have a coincidence point in $\Re$.

Solution: Since $(\Re,\delta)$ is a complete metric space endowed with partial order. Let $\alpha_0\in\Re$ such that $\xi\alpha_0\le\zeta\alpha_0$ and also note that ζ and ξ are continuous, ζ is monotone ξ- non decreasing and $\zeta(\alpha)\subseteq\xi(\alpha)$ for all $\alpha\in\Re$.
Now, choose $\alpha,\nu\in\Re$ such that $\alpha<\nu$, then

$$\delta(\zeta\alpha,\zeta\nu)=|\zeta\alpha-\zeta\nu|=|\frac{\alpha^2}{3}-\frac{\nu^2}{3}|=\frac{1}{3}(\alpha+\nu)|\alpha-\nu|. \tag{3.35}$$

$$\delta(\xi\alpha,\xi\nu)=|\xi\alpha-\xi\nu|=|\frac{3\alpha^2}{1+\alpha}-\frac{3\nu^2}{1+\nu}|=3|\frac{\alpha^2}{1+\alpha}-\frac{\nu^2}{1+\nu}|$$

$$=3|\frac{\alpha^2+\alpha^2\nu-\nu^2-\alpha\nu^2}{(1+\alpha)(1+\nu)}|=\frac{3}{(1+\alpha)(1+\nu)}|(\alpha-\nu)(\alpha+\nu)+\alpha\nu(\alpha-\nu)|$$

$$=\frac{3}{(1+\alpha)(1+\nu)}(\alpha+\nu+\alpha\nu)|\alpha-\nu|. \tag{3.36}$$

From equations (3.17) and (3.18), we have

$$\delta(\zeta\alpha,\zeta\nu)\le\delta(\xi\alpha,\xi\nu) \tag{3.37}$$

for all $\alpha,\nu\in\Re$. Again

$$\delta(\xi\alpha,\zeta\alpha)=|\xi\alpha-\zeta\alpha|=|\frac{3\alpha^2}{1+\alpha}-\frac{\alpha^2}{3}|=\frac{\alpha^2(8-\alpha)}{3(1+\alpha)}. \tag{3.38}$$

Similarly, we have

$$\delta(\xi\nu,\zeta\nu)=\frac{\nu^2(8-\nu)}{3(1+\nu)}. \tag{3.39}$$

Now, we calculate

$$\frac{\delta(\xi\alpha,\zeta\alpha)\delta(\xi\nu,\zeta\nu)}{\delta(\xi\alpha,\xi\nu)}=\frac{\alpha^2\nu^2(8-\alpha)(8-\nu)}{27(\alpha+\nu+\alpha\nu)|\alpha-\nu|} \tag{3.40}$$

$$\frac{\delta(\xi\alpha,\zeta\alpha)[\delta(\xi\nu,\zeta\nu)+1]}{\delta(\xi\alpha,\xi\nu)+1}=\frac{\alpha^2(8-\alpha)\{3(1+\nu)+\nu^2(8-\nu)\}}{9(1+\alpha)(1+\nu)+3(\alpha+\nu+\alpha\nu)|\alpha-\nu|} \tag{3.41}$$

From above equations, we have

$$\begin{aligned}\frac{1}{3}(\alpha+\nu)|\alpha-\nu|\le\mathbb{K}[&\frac{\alpha^2\nu^2(8-\alpha)(8-\nu)}{27(\alpha+\nu+\alpha\nu)|\alpha-\nu|}]\\ &+\mathbb{L}[\frac{\alpha^2(8-\alpha)\{3(1+\nu)+\nu^2(8-\nu)\}}{9(1+\alpha)(1+\nu)+3(\alpha+\nu+\alpha\nu)|\alpha-\nu|}]\\ &+\mathbb{Q}[\frac{3}{(1+\alpha)(1+\nu)}(\alpha+\nu+\alpha\nu)|\alpha-\nu|].\end{aligned} \tag{3.42}$$

Therefore

$$\delta(\zeta\alpha,\zeta\nu) \leq \mathbb{K}[\frac{\delta(\xi\alpha,\zeta\alpha)\delta(\xi\nu,\zeta\nu)}{\delta(\xi\alpha,\xi\nu)}] + \mathbb{L}[\frac{\delta(\xi\alpha,\zeta\alpha)[\delta(\xi\nu,\zeta\nu)+1]}{\delta(\xi\alpha,\xi\nu)+1}]$$
$$+\mathbb{Q}[\delta(\xi\alpha,\xi\nu)]$$

is satisfied. And hence the contraction condition defined in theorems (3.1) and (3.2) satisfied for all $\alpha, \nu \in \Re$. Now, we calculate coincidence point of ζ and ξ, such that $\zeta\alpha = \xi\alpha$,

$$\zeta(\alpha) = \xi(\alpha) \Rightarrow \frac{\alpha^2}{3} = \frac{3\alpha^2}{1+\alpha} \Rightarrow \alpha = 0,8. \tag{3.43}$$

Hence 0 and 8 are coincidence points of ζ and ξ.
Also, we calculate fixed point of ζ and ξ, such that $\zeta\alpha = \alpha$ and $\xi\alpha = \alpha$

$$\zeta(\alpha) = \alpha \Rightarrow \frac{\alpha^2}{3} = \alpha \Rightarrow \alpha = 0,3. \tag{3.44}$$

Hence 0 and 3 are fixed points of ζ. Similarly,

$$\xi(\alpha) = \alpha \Rightarrow \frac{3\alpha^2}{1+\alpha} = \alpha \Rightarrow \alpha = 0,\frac{1}{2}. \tag{3.45}$$

Thus 0 and $\frac{1}{2}$ are fixed points of ξ.
Therefore 0 is only common fixed point of ζ and ξ.
Also, 0 is only common fixed point and coincidence point of ζ and ξ.

□

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